$$I = \int_0^a \left[\int_0^{a^2 - y^2} \sqrt{(a^2 - y^2) - x^2} \cdot dx \right] .dy$$

Using the formula,

$$\int \sqrt{a^2 - x^2} \cdot dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a};\ \text{we get}$$

$$I = \int_0^a \left[\frac{x}{2} \cdot \sqrt{a^2 - y^2 - x^2} + \frac{a^2 - y^2}{2} \cdot \sin^{-1} \frac{x}{\sqrt{a^2 - y^2}} \right]_0^{\sqrt{a^2 - y^2}}$$

$$= \int_0^a \left[\frac{a^2 - y^2}{2} \cdot \sin^{-1}(1) \right] dy$$

$$= \frac{\pi}{4} \int_0^a (a^2 - y^2) \cdot dy = \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a = \frac{\pi a^3}{6}$$

$$I = \int_0^1 \frac{dy}{\sqrt{1 - y^2}} \int_0^1 \frac{dx}{\sqrt{1 - x^2}}$$

$$= \left[\sin^{-1} y \right]_0^1 \left[\sin^{-1} x \right]_0^1 = \frac{\pi}{2} . \frac{\pi}{2} = \frac{\pi^2}{4}$$

3) Let $I = \int_0^1 dy \left[\int_0^{\sqrt{\frac{1}{2}(1-y^2)}} \frac{dx}{\sqrt{(1-y^2)-x^2}} \right]$

$$= \int_0^1 dy \left[\sin^{-1} \frac{x}{\sqrt{1-y^2}} \right]_0^{\sqrt{\frac{1}{2}(1-y^2)}}$$

$$= \int_0^1 dy \left[\sin^{-1} \frac{1}{\sqrt{2}} - 0 \right] = \frac{\pi}{4} \int_0^1 dy = \frac{\pi}{4}$$

4) $\therefore \quad I = \int_0^1 dy \left[\int_0^\infty e^{-x^a \cdot y} \, dx \right]$

Let $x^a \cdot y = t$

$\therefore \quad x = \frac{t^{1/a}}{y^{1/a}} \qquad \therefore \quad dx = \frac{\frac{1}{a} t^{\frac{1}{a}-1}}{y^{1/a}} dt$

x	0	∞
t	0	∞

$\therefore \quad I = \int_0^1 dy \left[\int_0^\infty e^{-t} \cdot t^{\frac{1}{a}-1} \cdot \frac{1}{a\, y^{1/a}} \cdot dt \right]$

$$\int_0^{\infty} e^{-x} \cdot x^{n-1} \cdot dx = \Gamma(n)$$

$$= \frac{1}{a} \int_0^1 y^{-\frac{1}{a}} \cdot dy \int_0^{\infty} e^{-t} \cdot t^{\frac{1}{a}-1} \cdot dt$$

$$= \frac{1}{a} \left[\frac{y^{-\frac{1}{a}+1}}{-\frac{1}{a}+1} \right]_0^1 \Gamma\left(\frac{1}{a}\right) = \frac{\Gamma\left(\frac{1}{a}\right)}{a-1} \qquad (a > 1)$$

Ex. 5 : Evaluate $\int_{-1}^{1}\int_{0}^{1-x} x^{1/3}\, y^{-1/2}\,(1-x-y)^{1/2}\,dx\,dy.$

Let $y = (1-x)t$

Sol. : Let $1 - x = a$

$$\therefore \quad I_1 = \int_0^{1-x} y^{-1/2}(1-x-y)^{1/2} = \int_0^a y^{-/2}(a-y)^{1/2}\,dy$$

Now, put $\quad y = a\,t,\ dy = a\,d\,t.$

$$\therefore \quad I_1 = \int_0^1 a^{-1/2}\,t^{-1/2}\,a^{1/2}(1-t)^{1/2}\,a\ dt$$

$$= a\int_0^1 t^{-1/2}(1-t)^{1/2}\,d\,t = a\,B\left(\frac{1}{2},\frac{3}{2}\right) = a\,\frac{\Gamma(1/2)\ \Gamma(3/2)}{\Gamma(2)}$$

$$= a\,\frac{\Gamma(1/2)\cdot(1/2)\,\Gamma(1/2)}{1!} = \frac{\pi\,a}{2}$$

$$\therefore \quad I = \int_{-1}^{1} x^{1/3}\,\frac{\pi\,a}{2}\,dx = \frac{\pi}{2}\int_{-1}^{1} x^{1/3}(1-x)\,dx$$

$$= \frac{\pi}{2}\int_{-1}^{1}(x^{1/3} - x^{4/3})\,dx = \frac{\pi}{2}\left[\frac{x^{4/3}}{4/3} - \frac{x^{7/3}}{7/3}\right]_{-1}^{1}$$

$$= \frac{\pi}{2}\left[\frac{3}{4}(1) - \frac{3}{7}(1) - \frac{3}{4}(1) + \frac{3}{7}(-1)\right]$$

$$= \frac{\pi}{2}\left[-\frac{6}{7}\right] = -\frac{3\pi}{7}.$$

(6) $I = \int_0^1 \int_0^{\sqrt{a^2+x^2}} \frac{dy}{(a^2+x^2)+y^2}\,dx$

$$= \int_0^1 \left[\frac{1}{\sqrt{a^2+x^2}} \tan^{-1} \frac{y}{\sqrt{a^2+x^2}}\right]_0^{\sqrt{a^2+x^2}} dx$$

$$= \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{a^2+x^2}} \left(\frac{\pi}{4} - 0\right) dx = \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{a^2+x^2}}$$

$$= \frac{\pi}{4}\left[\log\left(x + \sqrt{x^2+a^2}\right)\right]_0^1$$

$$= \frac{\pi}{4}\left[\log\left(1 + \sqrt{a^2+1}\right) - \log(a)\right]$$

$$= \frac{\pi}{4}\left[\log\left(\frac{1+\sqrt{a^2+1}}{a}\right)\right]$$

(87) $\int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2 y^2 \, dx \, dy.$ **(M.U. 1987, 91, 2003)**

.: We have $x = \pm\sqrt{2-y}$ $\therefore$ $x^2 = 2 - y$ *i.e.* $y - 2 = -x^2$.

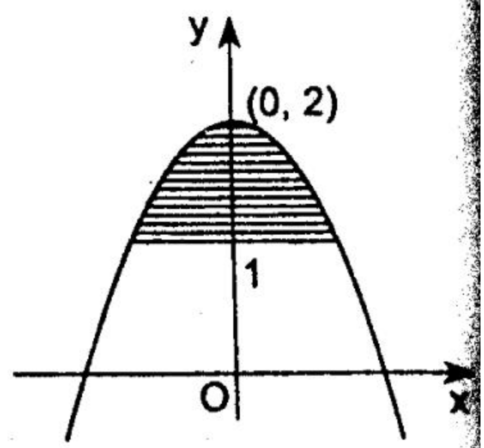
The curve is a parabola with vertex at (0, 2) as shown in the figure. And y es from 1 to 2.

$$\therefore \quad I = \int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2 y^2 \, dx \, dy$$

$$= 2\int_1^2 \int_0^{\sqrt{2-y}} 2x^2 y^2 \, dx \, dy$$

$$= 2\int_1^2 2y^2\left[\frac{x^3}{3}\right]_0^{\sqrt{2-y}} dy$$

$$= \frac{4}{3}\int_1^2 y^2 (2-y)^{3/2}\, dy$$



Putting $2 - y = t$, $dy = -dt$.

When $y = 1, t = 1$; when $y = 2$, $t = 0$.

$$\therefore \quad I = \frac{4}{3}\int_1^0 -(2-t)^2 \cdot t^{3/2}\, dt$$

$$= \frac{4}{3}\int_0^1 (4 - 4t + t^2)\, t^{3/2}\, dt$$

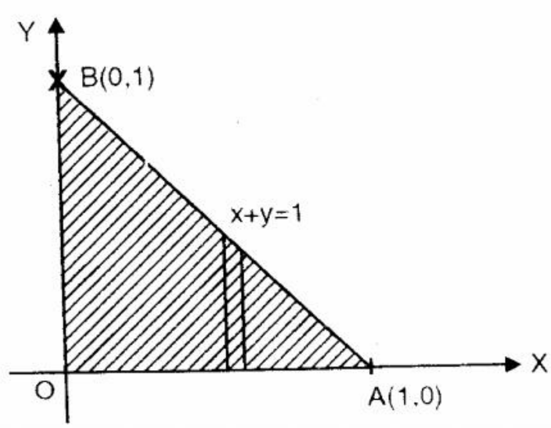
$$= \frac{4}{3}\int_0^1 (4t^{3/2} - 4t^{5/2} + t^{7/2})\, dt$$

$$= \frac{4}{3}\left[4 \cdot \frac{2}{5}t^{5/2} - 4 \cdot \frac{2}{7}t^{7/2} + \frac{2}{9}t^{9/2}\right]_0^1$$

$$= \frac{4}{3}\left[\frac{8}{5} - \frac{8}{7} + \frac{2}{9}\right] = \frac{856}{945}.$$

Lower and upper ends of the strip are at $y = 0$ and $y = 1 - x$; and

Left and right ends of the region are $x = 0$ and $x = 1$.



$$\therefore I = \int_0^1 \int_1^{1-x} \sqrt{xy\,(1-x-y)} \cdot dx \cdot dy$$

$$= \int_0^1 \sqrt{x}\left[\int_0^{1-x} \sqrt{y}\,\sqrt{(1-x)-y} \cdot dy\right] \cdot dx$$

Let $y = (1 - x)\, t$

$\therefore \quad dy = (1 - x)\, dt \qquad (\because x \text{ constant})$

and

y	0	1 – x
t	0	1

$$\therefore \quad I = \int_0^1 \sqrt{x} \left[\int_0^1 \sqrt{(1-x)\,t}\, \sqrt{(1-x) - (1-x)\,t}\, (1-x)\, dt \right] dx$$

$$= \int_0^1 \sqrt{x}\, (1-x)^2 \cdot dx \int_0^1 \sqrt{t}\, \sqrt{1-t} \cdot dt$$

($\because$ Limits are constants and variables can be separated)

$$= B\left(\frac{3}{2}, 3\right) \cdot B\left(\frac{3}{2}, \frac{3}{2}\right) \qquad \left(\because \int_0^1 x^{m-1} \cdot (1-x)^{n-1} \cdot dx = B(m, n) \right)$$

$$= \frac{\left\lceil \frac{3}{2} \right. \lceil 3}{\left\lceil \frac{9}{2} \right.} \cdot \frac{\left\lceil \frac{3}{2} \right. \left\lceil \frac{3}{2} \right.}{\lceil 3}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}\, \frac{1}{2}\sqrt{\pi}\, \frac{1}{2}\sqrt{\pi}}{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}} \qquad \left(\because \lceil n+1 = n \lceil n \text{ and } \left\lceil \frac{1}{2} \right. = \sqrt{\pi} \right)$$

$$= \frac{2\pi}{105}$$

Step I : At A, we have

(89)

$$x^4 = -x$$

$$\therefore \quad x^4 + x = 0$$

$$\therefore \quad x = 0, -1$$

$$\text{and} \quad y = 0, 1$$

$\therefore$ The points of intersection are O (0, 0) and A (– 1, 1)

We choose the strip say, parallel to X-axis.

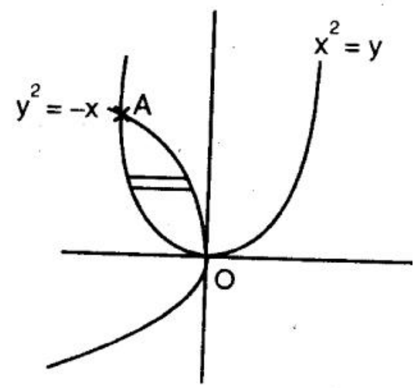


Fig. 9.5

Left and right ends of the strip are at $x = -\sqrt{y}$ and $x = -y^2$

And lower and upper values are $y = 0$ and $y = 1$

$$\therefore \quad I = \int_0^1 \int_{-\sqrt{y}}^{-y^2} xy \, dx \, dy$$

Step (II) :

$$\therefore \quad I = \int_0^1 y \, dy \left[\int_{-\sqrt{y}}^{-y^2} x \, dx \right]$$

$$= \int_0^1 y \, dy \left[\frac{x^2}{2} \right]_{-\sqrt{y}}^{-y^2} = \frac{1}{2} \int_0^1 y \, [y^4 - y] \, dy$$

$$= \frac{1}{2} \left[\frac{y^6}{6} - \frac{y^3}{3} \right]_0^1 = \frac{1}{2} \left(\frac{1}{6} - \frac{1}{3} \right) = -\frac{1}{12}$$

(Q10) Let $I = \iint\limits_R \frac{y\sqrt{x}}{\sqrt{(1-x^2)(1-y^2)}}\,dx \cdot dy$

We sketch the curves.

Since it is easier to integrate the inner integral w.r.t. y, we choose the strip parallel to Y-axis and we write,

$$I = \int_0^1 dx \int_0^{\sqrt{x}} \frac{y\sqrt{x}}{\sqrt{1-x^2}\,\sqrt{1-y^2}}\,dy$$

$$= -\frac{1}{2}\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x^2}}\,dx \int_0^{\sqrt{x}} \frac{(-2y)\,dy}{\sqrt{1-y^2}}$$

$$= -\frac{1}{2}\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x^2}}\,dx\left[\,2\sqrt{1-y^2}\,\right]_0^{\sqrt{x}}$$

$$= -\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x^2}}\,dx\left[\,\sqrt{1-x}-1\,\right]$$

$$= -\int_0^1 \frac{\sqrt{x}}{\sqrt{1+x}}\,dx + \int_0^1 \frac{\sqrt{x}}{\sqrt{1-x^2}}\,dx \quad .....(i)$$

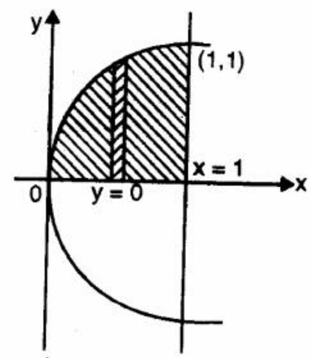


Fig. Ex. 9.2

Let $I_1 = \int_0^1 \frac{\sqrt{x}}{\sqrt{1+x}}\,dx$

Let $x = \tan^2\theta$

$\therefore \quad dx = 2\tan\theta\sec^2\theta\, d\theta$

x	0	1
θ	0	π / 4

$$\therefore \quad I_1 = \int_0^{\pi/4} \frac{\tan\theta}{\sec\theta}\, 2\tan\theta\cdot\sec^2\theta\cdot d\theta$$

$$= 2\int_0^{\pi/4} \tan^2\theta\cdot\sec\theta\cdot d\theta \quad .....(ii)$$

$$= 2\int_0^{\pi/4} (\sec^2\theta - 1)\sec\theta\cdot d\theta$$

$$= 2\int_0^{\pi/4} \sec^2\theta\cdot\sec\theta\cdot d\theta - 2\int_0^{\pi/4} \sec\theta\cdot d\theta$$

y parts,

$$= 2\left[(\tan\theta\sec\theta)_0^{\pi/4} - \int_0^{\pi/4} \tan\theta\cdot\sec\theta\tan\theta\, d\theta\right] - 2\left[\log(\sec\theta+\tan\theta)\right]_0^{\pi/4}$$

$$\therefore \quad I_1 = 2(\sqrt{2}) - 2\int_0^{\pi/4} \sec\theta \cdot \tan^2\theta \cdot d\theta - 2\log(\sqrt{2}+1)$$

$$= 2\sqrt{2} - I_1 - 2\log(\sqrt{2}+1)$$

from (ii)

$$\therefore \quad 2I_1 = 2\sqrt{2} - 2\log(\sqrt{2}+1)$$

$$\therefore \quad I_1 = \sqrt{2} - \log(\sqrt{2}+1) \qquad .....(iii)$$

Let $\quad I_2 = \int_0^1 \frac{\sqrt{x}}{\sqrt{1-x^2}}\,dx$

Let $\quad x = \sin\alpha$

$$\therefore \quad I_2 = \int_0^{\pi/2} \frac{\sqrt{\sin\alpha}}{\cos\alpha}\cos\alpha \cdot d\alpha = \int_0^{\pi/2} (\sin\alpha)^{1/2} \cdot (\cos\alpha)^0 \cdot d\alpha$$

$$= \frac{1}{2}B\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{1}{2}\;\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = 2\sqrt{\pi}\;\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$$

$$= \frac{2\sqrt{\pi}\left(\Gamma\left(\frac{3}{4}\right)\right)\left(\Gamma\left(\frac{1}{4}\right)\right)}{\left(\Gamma\left(\frac{1}{4}\right)\right)^2} = \frac{2\sqrt{\pi}}{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}\cdot\frac{\pi}{\sin\frac{\pi}{4}}$$

$$\left(\text{since } \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}}\right)$$

$$= \frac{2\sqrt{\pi}}{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}\cdot\frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$= \frac{2\sqrt{2}\,\pi\,\sqrt{\pi}}{\left(\Gamma\frac{1}{4}\right)^2}$$

Substituting (iii) and (iv) in (i), we get

$$I = -\left[\sqrt{2} - \log\left(\sqrt{2}+1\right)\right] + \frac{2\,\pi\,\sqrt{2\,\pi}}{\left(\Gamma\frac{1}{4}\right)^2}$$

$$= \log\left(\sqrt{2}+1\right) - \sqrt{2} + \frac{2\,\pi\,\sqrt{2\,\pi}}{\left(\Gamma\frac{1}{4}\right)^2}$$

(Q11) Let $I = \iint_R xy\,(x-1)\,dx\,dy$

The curve $xy = 4$ is a rectangular hyperbola.

The sketch is :

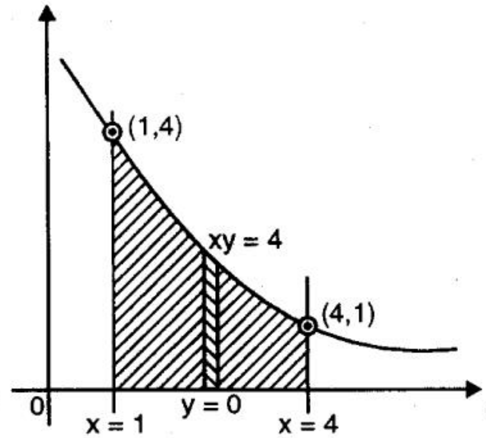


Fig. Ex. 9.5

Choosing the strip parallel to Y-axis

$$I = \int_1^4 x\,(x-1)\,dx \int_0^{4/x} y\,dy$$

$$= \int_1^4 x\,(x-1)\,dx \left[\frac{y^2}{2}\right]_0^{4/x}$$

$$= \frac{1}{2}\int_1^4 x\,(x-1)\,\frac{16}{x^2}\,dx = 8\int_1^4 \left(\frac{x-1}{x}\right) dx$$

$$= 8\,[\,x - \log x\,]_1^4 = 8\,[\,4 - \log 4 - 1\,]$$

$$= 8\,(3 - \log 4)$$

(Q12) Let $I = \iint\limits_R \frac{2xy^5}{\sqrt{1 + x^2 y^2 - y^4}}\, dx \cdot dy$

The sketch :

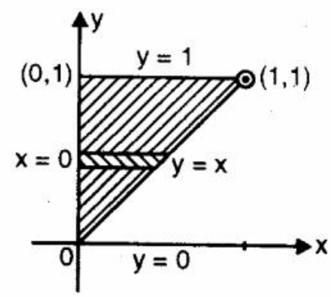


Fig. Ex. 9.8

We choose the strip parallel to X-axis as is convenient to integrate the inner integral w.r.t. - x.

$$\therefore \quad I = \int_0^1 y^5\, dy \int_0^y \frac{2x\,dx}{\sqrt{1 - y^4 + x^2 y^2}}$$

$$= \int_0^1 \frac{y^5}{y^2}\, dy \int_0^y \frac{2xy^2}{\sqrt{1 - y^4 + x^2 y^2}} = \int_0^1 y^3\, dy \left[2\sqrt{1 - y^4 + x^2 y^2} \right]_0^y$$

$$\left(\text{since } \int \frac{f'}{\sqrt{f}} = 2\sqrt{f} \right.$$

$$= 2\int_0^1 y^3 \left[\sqrt{1} - \sqrt{1 - y^4} \right] dy = 2\int_0^1 y^3\, dy - 2\int_0^1 y^3 \sqrt{1 - y^4} \cdot dy$$

$$= 2\left[\frac{y^4}{4} \right]_0^1 + \frac{1}{2}\int_0^1 (-4y^3)\sqrt{1 - y^4} \cdot dy = \frac{2}{4} + \frac{1}{2}\left[\frac{2}{3}(1 - y^4)^{3/2} \right]_0^1$$

$$= \frac{1}{2} + \frac{1}{3}(-1) = \frac{1}{6}$$

The sketch :

(Q13)

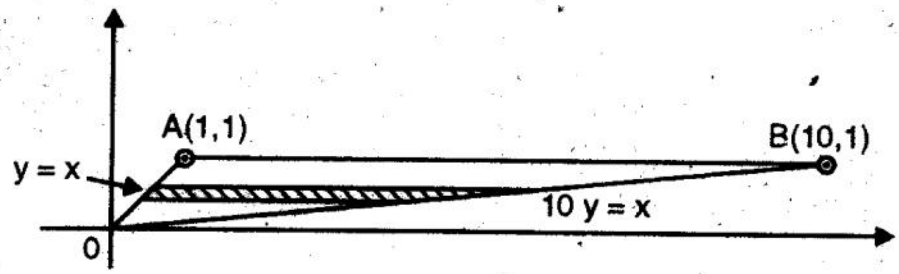


Fig. Ex. 9.19

$$I = \iint_S \sqrt{xy - y^2} \cdot dx \cdot dy$$

We take strip parallel to X-axis.

$$\therefore \quad I = \int_0^1 dy \int_y^{10y} \sqrt{y}\,\sqrt{x-y}\, dx = \int_0^1 \sqrt{y}\, dy \left[\frac{2}{3}(x-y)^{3/2}\right]_y^{10y}$$

$$= \frac{2}{3}\int_0^1 \sqrt{y}\left[(9y)^{3/2} - 0\right] dy = \frac{2}{3}\int_0^1 \sqrt{y}\,(27\,y^{3/2})\, dy$$

$$= 18\int_0^1 y^2\, dy = 18\left(\frac{y^3}{3}\right)_0^1 = 6$$

Soln. : Let $I = \iint xy\, dx \cdot dy$

(Q14) Note that $x^2 + y^2 - 2x = 0$ is a circle;

$y^2 = 2x$ is a parabola and

$y = x$ is a straight line.

The region is as shaded :

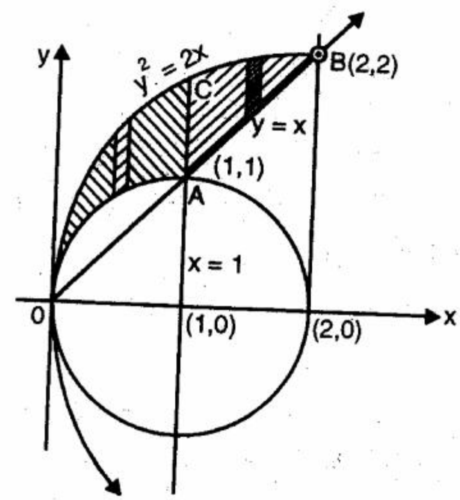


Fig. Ex. 9.21

Also, $x^2 + y^2 - 2x = 0$

$\therefore\ y^2 = 2x - x^2$

$\therefore\ y = \pm\sqrt{2x - x^2}$

in first quadrant, $y = +\sqrt{2x - x^2}$

We choose strip parallel to Y-axis.

$$\therefore\ I = \int_0^1 dx \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} xy\, dy + \int_1^2 dx \int_x^{\sqrt{2x}} xy\, dy$$

$$= \int_0^1 x\, dx \left[\frac{y^2}{2}\right]_{\sqrt{2x-x^2}}^{\sqrt{2x}} + \int_1^2 x\, dx \left[\frac{y^2}{2}\right]_x^{\sqrt{2x}}$$

$$= \frac{1}{2}\int_0^1 x\, dx \left[2x - (2x - x^2)\right] + \frac{1}{2}\int_1^2 x\, dx \left[2x - x^2\right]$$

$$= \frac{1}{2}\int_0^1 x^3 \cdot dx + \frac{1}{2}\int_1^2 (2x^2 - x^3)\, dx = \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1 + \frac{1}{2}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{8} + \frac{1}{2}\left[\frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4}\right] = \frac{1}{8} + \frac{1}{2}\left[\frac{14}{3} - \frac{15}{4}\right] = \frac{7}{12}$$

(Q15) Let $I = \iint_R x(x-y)\,dx \cdot dy$

Let O (0, 0), A (1, 2), B (0. 4) be the points.

Equation of OA is $y = 2x$

and slope of AB $= \dfrac{4-2}{0-1} = -2$

∴ Equation of AB is $y - 4 = -2(x-0)$

$\therefore \quad y + 2x = 4$

We take the strip parallel to Y-axis.

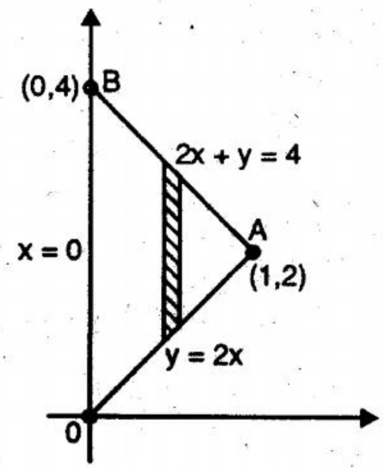


Fig. Ex. 9.25

$$\therefore \quad I = \int_0^1 x\,dx \int_{2x}^{4-2x} (x-y)\,dy$$

$$= \int_0^1 x\,dx \left[xy - \frac{y^2}{2} \right]_{2x}^{4-2x}$$

$$= \int_0^1 x \left[x(4-2x) - \frac{(4-2x)^2}{2} - x(2x) + \frac{4x^2}{2} \right] dx$$

$$= \int_0^1 \left[4x^2 - 2x^3 - 8x + 8x^2 - 2x^3 \right] dx$$

$$= \int_0^1 \left[12x^2 - 4x^3 - 8x \right] dx$$

$$= \left[4x^3 - x^4 - 4x^2 \right]_0^1 = 4 - 1 - 4 = -1$$

Soln. : We trace the region :

$$I = \iint_R \frac{y\, e^{2y} \cdot dx \cdot dy}{\sqrt{(1-x)(x-y)}}$$

We choose the strip parallel to X-axis.

$$\therefore \quad I = \int_0^1 y\, e^{2y} \cdot dy \int_y^1 \frac{dx}{\sqrt{(1-x)(x-y)}} \qquad .....(i)$$

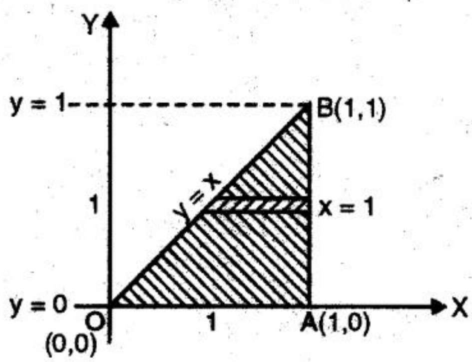


Fig. Ex. 9.29

Consider,
$$\frac{1}{\sqrt{(1-x)(x-y)}} = \frac{1}{\sqrt{-x^2 + x(y+1) - y}}$$

$$= \frac{1}{\sqrt{-x^2 + x(y+1) - \frac{(y+1)^2}{4} + \frac{(y+1)^2}{4} - y}}$$

$$= \frac{1}{\sqrt{\frac{(y-1)^2}{4} - \left[x - \frac{(y+1)}{2}\right]^2}}$$

from (i),

$$I = \int_0^1 y\, e^{2y} \cdot dy \int_y^1 \frac{1}{\sqrt{\left(\frac{y-1}{2}\right)^2 - \left[x - \left(\frac{y+1}{2}\right)\right]^2}}\, dx$$

$$= \int_0^1 y\, e^{2y} \cdot dy \left[\sin^{-1} \frac{x - \left(\frac{y+1}{2}\right)}{\left(\frac{y-1}{2}\right)}\right]_y^1$$

$$= \int_0^1 y\, e^{2y} \left[\left(-\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right)\right] dy = -\pi \int_0^1 y \cdot e^{2y} \cdot dy$$

$$= -\pi \left[y\left(\frac{e^{2y}}{2}\right) - (1)\left(\frac{e^{2y}}{4}\right)\right]_0^1$$

$$= -\pi\left[\frac{1}{2}e^2 - \frac{e^2}{4} + \frac{1}{4}\right] = -\pi\left[\frac{e^2}{4} + \frac{1}{4}\right] = \frac{-\pi}{4}\left(1 + e^2\right)$$

(817) Let $I = \iint_R x^2\, dx \cdot dy$

The curve $y = \frac{16}{x}$

i.e. $xy = 16$ is a rectangular hyperbola.

The points of intersection of $y = x$ and $xy = 16$ are ;

$$x^2 = 16$$

$$\therefore \quad x = \pm 4$$

$$y = \pm 4$$

$\therefore$ The points are (4, 4), (− 4 , − 4)

The sketch.

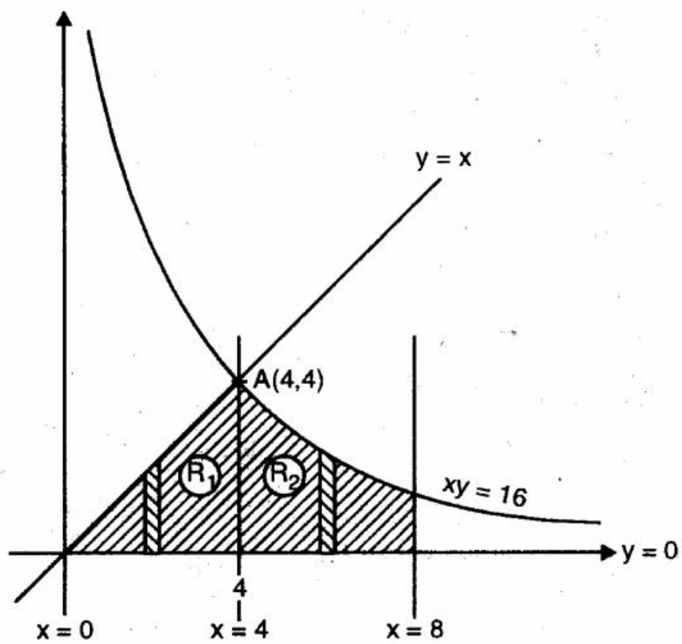


Fig. Ex. 9.38

We choose the strip parallel to Y-axis. We divide the region into two subregions.

$$\text{and} \quad I = \iint_R x^2\, dx\, dy = \int_0^4 \int_0^x x^2\, dx\, dy + \int_4^8 \int_0^{\frac{16}{x}} x^2\, dx\, dy$$

$$= \int_0^4 x^2\,dx\,(x) + \int_4^8 x^2\,dx\left(\frac{16}{x}\right)$$

$$= \left[\frac{x^4}{4}\right]_0^4 + 16\left[\frac{x^2}{2}\right]_4^8 = 64 + 8\,[\,64 - 16\,]$$

$$= 64 + 8 \times 48 = 64 + 384 = 448.$$

Q18) Let $I = \iint_R \frac{y}{(a-x)\sqrt{ax-y^2}}\,dx\,dy$

We sketch the curves :

Since it is easier to integrate inner integral w.r.t. y, we take strip parallel to Y-axis, as shown.

Hence limits are

$y = x$ to $y = \sqrt{x}$

and $x = 0$ to $x = 1$.

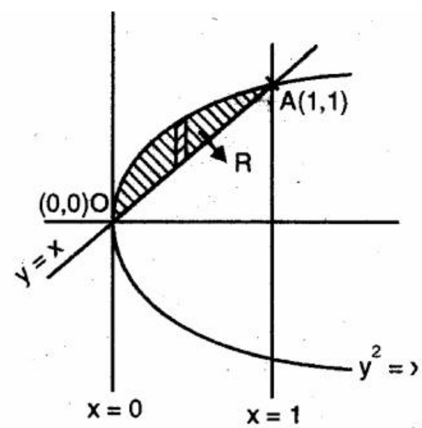


Fig. 9.44

$$\therefore \quad I = \int_0^1 \frac{dx}{(a-x)}\left[\int_x^{\sqrt{x}} \frac{y\,dy}{\sqrt{ax-y^2}}\right]$$

$$= \int_0^1 \frac{dx}{(a-x)} \left[-\sqrt{ax-y^2} \right]_x^{\sqrt{x}}$$

$$= \int_0^1 \frac{dx}{(a-x)} \left[-\sqrt{ax-x} + \sqrt{ax-x^2} \right] dx$$

$$= \int_0^1 \frac{dx}{(a-x)} \left[\sqrt{x}\,\sqrt{a-x} - \sqrt{x}\,\sqrt{a-1} \right] dx$$

$$= \int_0^1 \left[\sqrt{\frac{x}{a-x}} - \sqrt{a-1}\,\frac{\sqrt{x}}{(a-x)} \right] dx$$

Let $x = a\sin^2\theta$

$\therefore \quad dx = 2a\sin\theta\cos\theta \cdot d\theta$

x	0	1
θ	0	$\sin^{-1}\frac{1}{\sqrt{a}}$

$$= \int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \left[\frac{\sin\theta}{\cos\theta} - \sqrt{a-1} \cdot \frac{\sqrt{a}\sin\theta}{a\cos^2\theta} \right] 2a\sin\theta\cos\theta\, d\theta$$

$$= 2a \int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \left[\sin^2\theta - \frac{\sqrt{a-1}}{\sqrt{a}}\,\frac{\sin^2\theta}{\cos\theta} \right] d\theta$$

$$= 2a \int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \left[\frac{1-\cos 2\theta}{2} - \sqrt{\frac{a-1}{a}}\,(\sec\theta - \cos\theta) \right] d\theta$$

$$\left(\because \frac{\sin^2\theta}{\cos\theta} = \frac{1-\cos^2\theta}{\cos\theta} = \sec\theta - \cos\theta \right)$$

$$= 2a \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} - \sqrt{\frac{a-1}{a}} \left\{ \log(\sec\theta + \tan\theta) - \sin\theta \right\} \right]_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \quad \ldots (i)$$

if $\theta = \sin^{-1}\frac{1}{\sqrt{a}}$ then

$$\sin\theta = \frac{1}{\sqrt{a}} \text{ and}$$

$$\cos\theta = \sqrt{1-\frac{1}{a}} = \frac{\sqrt{a-1}}{\sqrt{a}}$$

$$\therefore \quad \sin 2\theta = 2\sin\theta\cos\theta = \frac{2\sqrt{a-1}}{a}$$

and
$$\sec\theta + \tan\theta = \frac{1+\sin\theta}{\cos\theta} = \frac{1+\frac{1}{\sqrt{a}}}{\frac{\sqrt{a-1}}{\sqrt{a}}} = \frac{\sqrt{a}+1}{\sqrt{a-1}}$$

Substituting in (i),

$$I = 2a\left[\frac{1}{2}\sin^{-1}\frac{1}{\sqrt{a}} - \frac{\sqrt{a-1}}{2a} - \sqrt{\frac{a-1}{a}}\log\left(\frac{\sqrt{a}+1}{\sqrt{a-1}}\right) + \sqrt{\frac{a-1}{a}}\cdot\frac{1}{\sqrt{a}}\right]$$

$$= 2a\left[\frac{1}{2}\sin^{-1}\frac{1}{\sqrt{a}} + \frac{\sqrt{a-1}}{2a} - \sqrt{\frac{a-1}{a}}\log\left(\frac{\sqrt{a}+1}{\sqrt{a-1}}\right)\right]$$

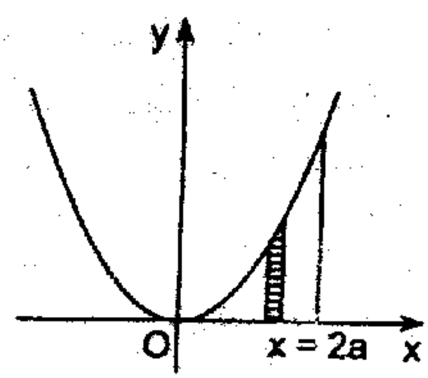
$$= a\sin^{-1}\frac{1}{\sqrt{a}} + \sqrt{a-1} - 2\sqrt{a(a-1)}\log\left(\frac{\sqrt{a}+1}{\sqrt{a-1}}\right)$$

(Q19)
$$I = \int_0^{2a}\int_0^{x^2/4a} xy\,dx\,dy$$

$$= \int_0^{2a}\left[x\cdot\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

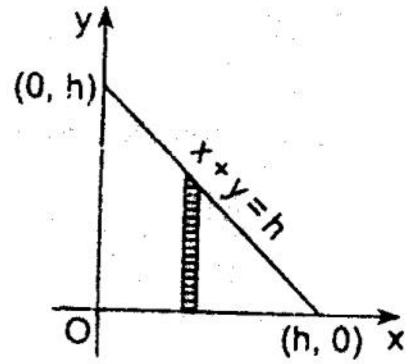
$$= \int_0^{2a}\frac{x}{2}\cdot\frac{x^4}{16a^2}\,dx = \frac{1}{32a^2}\int_0^{2a} x^5\,dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{1}{32a^2}\cdot\frac{64a^6}{6} = \frac{a^4}{3}.$$



(20) **Sol. :** The region is bounded by the x-axis, y-axis and the line $x + y = h$.

On the strip y varies from 0 to $h - x$ and then strip moves from $x = 0$ to $x = h$.



$$\therefore \quad I = \int_0^h \int_0^{h-x} x^{m-1} y^{n-1}\, dy\, dx$$

Let $\quad I_1 = \int_0^{h-x} y^{n-1}\, dy = \left[\frac{y^n}{n}\right]_0^{h-x}$

$$= \frac{1}{n}(h - x)^n$$

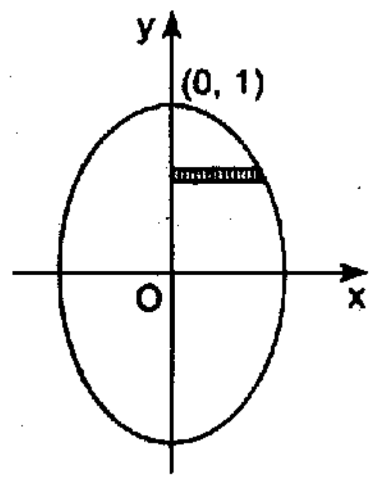
Now, $\quad I = \int_0^h x^{m-1} \cdot \frac{1}{n}(h-x)^n\, dx. \qquad$ Put $x = ht$

$$= \int_0^1 h^{m-1} \cdot t^{m-1} \cdot \frac{1}{n} h^n (1-t)^n \cdot h\, dt$$

$$= \frac{h^{m+n}}{n} \int_0^1 t^{m-1}(1-t)^n\, dt$$

$$= \frac{h^{m+n}}{n} \cdot \frac{\overline{|m}\ \overline{|n+1}}{|m+n+1} = \frac{h^{m+n}\ \overline{|m}\ \overline{|n}}{(m+n)\ |m+n}.$$

(21) **Sol. :** The ellipse $2x^2 + y^2 = 1$ i.e. $\frac{x^2}{1/2} + \frac{y^2}{1} = 1$ has semi-major axis $a = \frac{1}{\sqrt{2}}$ and semi-minor axis $b = 1$.



If we consider a strip parallel to the x-axis, on this strip x varies from $x = 0$ to $x = \sqrt{1 - y^2}\,/\sqrt{2}$. This strip moves from $y = 0$ to $y = 1$.

$$\therefore \quad I = \int_0^1 \int_0^{\sqrt{(1-y^2)/2}} \frac{dx\, dy}{\sqrt{(1-y^2) - x^2}}.$$

$$= \int_0^1 \sin^{-1}\left[\frac{x}{\sqrt{1-y^2}}\right]_0^{\sqrt{(1-y^2)/2}} dy$$

$$= \int_0^1 \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) dy = \int_0^1 \frac{\pi}{4} dy$$

$$= \frac{\pi}{4}[y]_0^1 = \frac{\pi}{4}.$$

$$I = \int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{dx \cdot dy}{(1 + e^y)\sqrt{a^2 - x^2 - y^2}}$$

Sol. : As it is, order of integration is with respect to y first and then with respect to x. Integrating with respect to y first is very complicated. So we change the order of integration.

Step (I) : Limits are :

(823) ||

$y = 0$ to $y = \sqrt{a^2 - x^2}$ i.e. $x^2 + y^2 = a^2$

$x = 0$ to $x = a$

We observe that region of integration is positive quadrant of positive circle

$$x^2 + y^2 = a^2$$

By changing the order,

$$I = \int_0^a \int_0^{\sqrt{a^2-y^2}} \frac{dx \cdot dy}{(1+e^y)\sqrt{a^2-x^2-y^2}}$$

$$I = \int_0^a \frac{dy}{1+e^y} \cdot \int_0^{\sqrt{a^2-y^2}} \frac{dx}{\sqrt{(a^2-y^2)-x^2}}$$

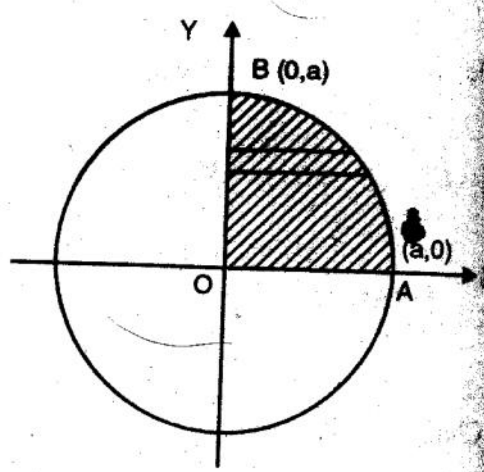


Fig. 9.13

$$= \int_0^a \frac{1}{1+e^y}\left\{\sin^{-1}\frac{x}{\sqrt{a^2-y^2}}\right\}_0^{\sqrt{a^2-y^2}} \cdot dy = \int_0^a \frac{1}{1+e^y}\left\{\frac{\pi}{2}-0\right\} dy$$

$$= \frac{\pi}{2}\int_0^a \frac{1+e^y-e^y}{1+e^y}\,dy = \frac{\pi}{2}\left[\int_0^a dy - \int_0^a \frac{e^y}{1+e^y}\,dy\right]$$

$$= \frac{\pi}{2}\left[a - \left\{\log(1+e^y)\right\}_0^a\right] \qquad \therefore \int \frac{f'}{f} = \log f$$

$$= \frac{\pi}{2}\left[a - \left\{\log(1+e^a) - \log 2\right\}\right] = \frac{\pi}{2}\left[a - \log\left(\frac{1+e^a}{2}\right)\right]$$

$$= \frac{\pi}{2}\left[\log e^a - \log\left(\frac{1+e^a}{2}\right)\right] = \frac{\pi}{2}\left[\log\frac{2e^a}{1+e^a}\right]$$

Q $I = \int_{0} \int_{0} \cos 2y \cdot \sqrt{1 - a^2 \sin^2 x} \cdot dx \cdot dy$

Sol. :

Here it is not possible to integrate with respect to x the inner integral. So we change the order of integration.

Step (I) : Given limits are :

(Q24) $\Rightarrow$ $x = 0$ to $x = y$

$y = 0$ to $y = \frac{\pi}{2}$

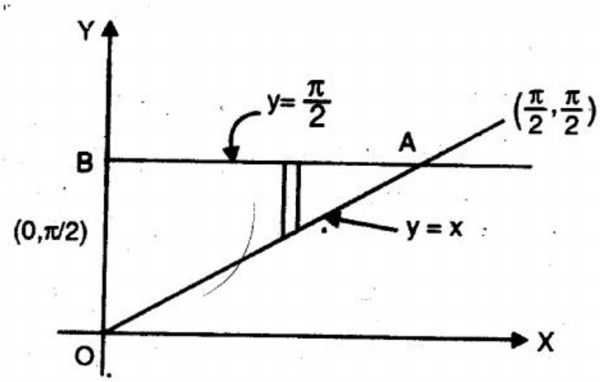


Fig. 9.14

Changing the order
$$I = \int_0^{\pi/2}\int_x^{\pi/2} \cos 2y \sqrt{1-a^2\sin^2 x}\cdot dx\cdot dy$$

$$= \int_0^{\pi/2} \sqrt{1-a^2\sin^2 x}\cdot\left[\int_x^{\pi/2} \cos 2y\cdot dy\right]dx$$

$$= \int_0^{\pi/2} \sqrt{1-a^2\sin^2 x}\cdot\left[\frac{\sin 2y}{2}\right]_x^{\pi/2}\cdot dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \sqrt{1-a^2\sin^2 x}\,(0-\sin 2x)\cdot dx \qquad \text{... (1)}$$

Let $1 - a^2\sin^2 x = t^2$

x	0	π/2
t	1	$\sqrt{1-a^2}$

$\therefore \quad -2a^2 \sin x\cdot\cos x\cdot dx = 2t\cdot dt$

i.e. $\quad \sin 2x\cdot dx = -\frac{2t}{a^2}dt$

$\therefore$ From above equation,

$$I = \frac{1}{2}\int_1^{\sqrt{1-a^2}} \sqrt{t^2}\cdot\left(+\frac{2t}{a^2}\right)\cdot dt = \frac{2}{a^2}\,\frac{1}{2}\left[\frac{t^3}{3}\right]_1^{\sqrt{1-a^2}}$$

$$= \frac{2}{3a^2}\cdot\frac{1}{2}\left[(1-a^2)^{3/2}-1\right] = \frac{1}{3a^2}\left[1-(1-a^2)^{3/2}\right]$$

$$= \frac{1}{3a^2}\left[(1-a^2)^{3/2}-1\right]$$

Sol. : (Q25)

As it is convenient to integrate the inner integral with respect to x, we change the order of integration.

Note that when all the limits are constant, order can be changed directly without plotting the curves.

$$\therefore \quad I = \int_1^{\infty} dy \int_0^1 e^{-y} \cdot y^x \cdot \log y \cdot dx = \int_1^{\infty} e^{-y} \cdot \log y \cdot dy \left[\int_0^1 y^x \cdot dx \right]$$

$$= \int_1^{\infty} e^{-y} \log y \left[\frac{y^x}{\log y} \right]_0^1 \cdot dy = \int_1^{\infty} e^{-y} (y-1) \cdot dy$$

Integrate by parts, $= \left[(y-1)(-e^{-y}) - (1)(e^{-y}) \right]_1^{\infty}$

$$= \frac{1}{e} \qquad (\because e^{-\infty} = 0)$$

Soln. (826)

Let $I = \int_0^1 dy \int_0^{\sqrt{1-x^2}} \frac{\cos^{-1} x \cdot dx}{\sqrt{(1-x^2-y^2)(1-x^2)}}$

To evaluate the integral, we change the order of integration.

Limits of integration $y = 0$ to $y = \sqrt{1-x^2}$ $(x^2+y^2=1)$ and $x = 0$ to $x = 1$.

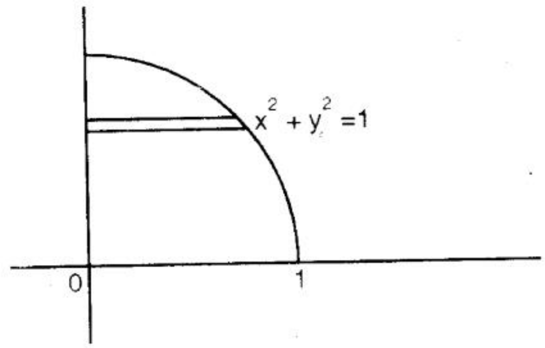


Fig. Ex. 2

Region of integration is given by,

$$\text{Hence,}\quad I = \int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{\cos^{-1} x \cdot dy}{\sqrt{(1-x^2-y^2)(1-x^2)}}$$

$$= \int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}}\, dx \int_0^{\sqrt{1-x^2}} \frac{dy}{\sqrt{(1-x^2)-y^2}}$$

$$= \int_0^1 \frac{\cos^{-1} x \cdot dx}{\sqrt{1-x^2}} \left[\sin^{-1} \frac{y}{\sqrt{1-x^2}}\right]_0^{\sqrt{1-x^2}}$$

$$= \int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}}\, dx \left(\frac{\pi}{2}\right) = \frac{\pi}{2}\int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}}\, dx$$

Let $x = \cos\theta,\ dx = -\sin\theta\, d\theta$

$$= \frac{\pi}{2}\int_{\pi/2}^{0} \frac{\theta}{\sin\theta}\cdot(-\sin\theta)\, d\theta = \frac{\pi}{2}\int_0^{\pi/2} \theta \cdot d\theta = \frac{\pi}{2}\cdot\frac{\pi^2}{8} = \frac{\pi^3}{16}$$

(Q27)

$y = 0$ to $y = x$

and $x = 0$ to $x = a$.

The region of integration is :

The given strip is parallel to Y-axis. We take the strip parallel to X-axis.

$$I = \int_0^a \int_y^a \frac{\sin y \cdot dx \cdot dy}{\sqrt{(a-x)(x-y)(4-5\cos y)^2}}$$

$$= \int_0^a \frac{\sin y \cdot dy}{\sqrt{(4-5\cos y)^2}} \int_y^a \frac{dx}{\sqrt{(a-x)(x-y)}}$$

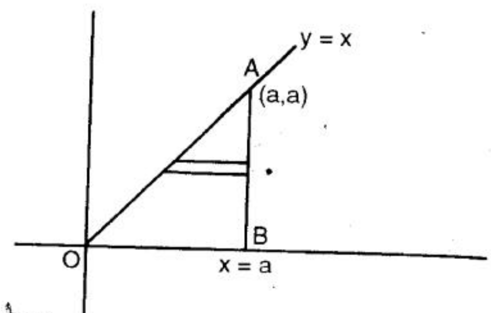


Fig. Ex. 3

For the inner integral, we substitute,

$$x - y = t^2$$

$$dx = 2t\ dt$$

Or Use Beta & Gamma Subs

$$\therefore \quad I = \int_0^a \frac{\sin y \cdot dy}{\sqrt{(4-5\cos y)^2}} \left[\int_0^{\sqrt{a-y}} \frac{2t\ dt}{t\sqrt{(a-y)-t^2}} \right]$$

$$= 2\int_0^a \frac{\sin y \cdot dy}{\sqrt{(4-5\cos y)^2}} \left[\sin^{-1} \frac{t}{\sqrt{a-y}} \right]_0^{\sqrt{a-y}} = 2\int_0^a \frac{\sin y}{\sqrt{(4-5\cos y)^2}} \cdot \frac{\pi}{2} \cdot dy$$

$$= \pi \int_0^a \frac{\sin y}{\pm(4-5\cos y)} dy = \pm\, \pi \int_0^a \frac{\sin y}{5\cos y - 4} dy$$

$$= \pm \pi \left[\log(5\cos y - 4) \right]_0^a = \pm \pi \left[\log(5\cos a - 4) \right]$$

Soln. :

(Q28) Let $I = \int_0^a \int_0^x \frac{e^y}{\sqrt{(a-x)(x-y)}} dx \cdot dy$

Since it is not convenient to integrate the inner integral w.r.t. y, we change the order of integration.

Given limits are :

$y = 0$ to $y = x$

and $x = 0$ to $x = a$.

To change the order, we take strip parallel to X-axis and we have

$$I = \int_0^a \int_y^a \frac{e^y}{\sqrt{(a-x)(x-y)}} dx\, dy$$

$$= \int_0^a e^y \cdot dy \int_y^a \frac{1}{\sqrt{(a-x)(x-y)}} dx$$

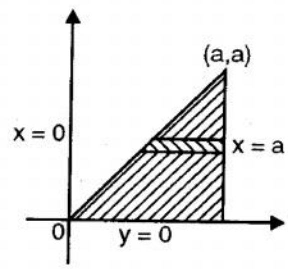


Fig. Ex. 9.3

$(x-y) = (a-y)t$

Let $x - y = t^2$

$\therefore \quad dx = 2t\, dt$

x	y	a
t	0	$\sqrt{a-y}$

$$\therefore \quad I = \int_0^a e^y \cdot dy \int_0^{\sqrt{a-y}} \frac{2t\,dt}{\sqrt{(a-y-t^2)\,t^2}} = 2\int_0^a e^y\, dy \int_0^{\sqrt{a-y}} \frac{dt}{\sqrt{(a-y)-t^2}}$$

$$= 2\int_0^a e^y\, dy \left[\sin^{-1}\frac{t}{\sqrt{a-y}}\right]_0^{\sqrt{a-y}} = 2\int_0^a e^y\, dy\left(\frac{\pi}{2}\right) = \pi\left(e^y\right)_0^a = \pi\,(e^a - 1)$$

Soln. :

Let $I = \int_0^{\pi} \int_0^{x} \frac{\sin y \cdot dy\, dx}{\sqrt{(\pi - x)(x - y)}}$

limits are : $y = 0$ to $y = x$

and $x = 0$ to $x = \pi$

and strip is parallel to Y-axis.

Points of intersection are $(0, 0), (\pi, \pi)$.

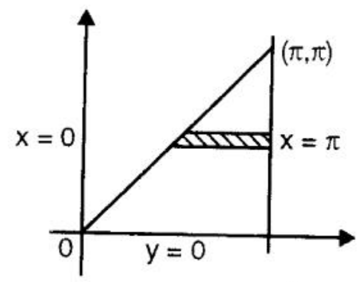


Fig. Ex. 9.24

To change the order, we take strip parallel to X-axis.

$$\therefore \quad I = \int_0^{\pi} \sin y \cdot dy \int_y^{\pi} \frac{dx}{\sqrt{(\pi - x)(x - y)}}$$

Let $x - y = t^2$

$\therefore \quad dx = 2t\, dt$

x	y	π
t	0	$\sqrt{\pi - y}$

$$\therefore \quad I = \int_0^{\pi} \sin y \cdot dy \int_0^{\sqrt{\pi - y}} \frac{2t\, dt}{\sqrt{\left[(\pi - y) - t^2\right] t^2}}$$

$$= 2\int_0^{\pi} \sin y \cdot dy \left[\sin^{-1} \frac{t}{\sqrt{\pi - y}}\right]_0^{\sqrt{\pi - y}}$$

$$= 2\int_0^{\pi} \sin y \cdot dy \left(\frac{\pi}{2}\right) = 2 \cdot \frac{\pi}{2}(-\cos y)_0^{\pi} = 2\pi$$

Soln. : (830)

Let $I = \int_{0}^{1}\int_{4y}^{4} e^{x^2} \cdot dx \cdot dy$

We change the order of integration, since inner integral can easily be evaluated w.r.t. y.

Limits are : $x = 4y$ to $x = 4$

$y = 0$ to $y = 1$

The point of intersection of $x = 4y$ and $x = 4$ is (4, 1)

Taking strip parallel to Y-axis,

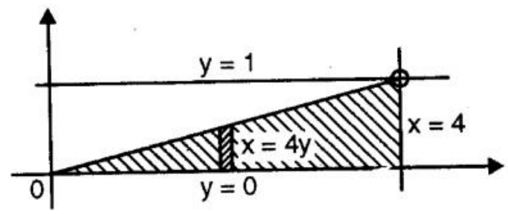


Fig. Ex. 9.27

$$I = \int_{0}^{4} dx \int_{0}^{x/4} e^{x^2} \cdot dy = \int_{0}^{4} e^{x^2} \cdot dx \left(\frac{x}{4}\right) dx$$

$$= \frac{1}{8}\int_{0}^{4} (2x)\, e^{x^2} \cdot dx = \frac{1}{8}\left[e^{x^2}\right]_0^4 = \frac{1}{8}\left[e^{16} - 1\right]$$

$$\left(\text{since } \int e^f \cdot f^I = e^f\right)$$

Soln. :

(Q31)

$$\text{Let, } I = \int_0^{1/2} \int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\,\sqrt{1-x^2-y^2}}\, dx\, dy$$

Given limits are,

$$x = 0 \quad \text{to} \quad x = \sqrt{1-4y^2}.$$

$$y = 0 \quad \text{to} \quad y = \frac{1}{2}$$

Given strip is parallel to X-axis.

$$\text{Also,} \quad x = \sqrt{1-4y^2}$$

$$\therefore \quad x^2 + 4y^2 = 1$$

$$\therefore \quad \frac{x^2}{1} + \frac{y^2}{1/4} = 1 \text{ is an ellipse}$$

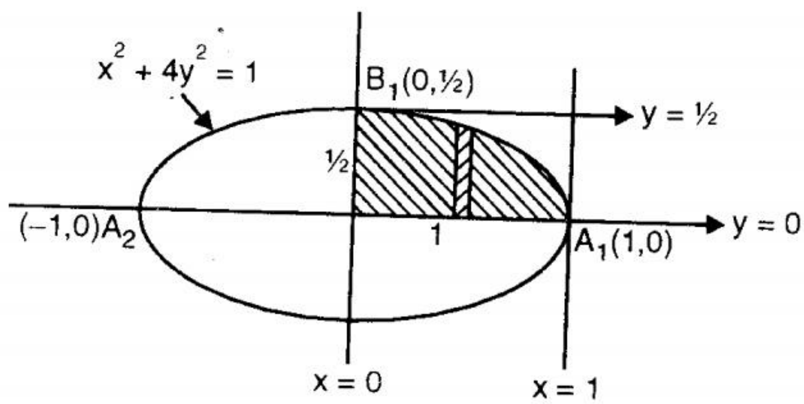


Fig. Ex. 9.30

When $y = \frac{1}{2}, x^2 + 4\frac{1}{4} = 1 \qquad \therefore \quad x = 0$

$\therefore \left(0, \frac{1}{2}\right)$ is the point of

We change the order of integration. We take strip parallel to Y-axis.

$$\therefore \quad I = \int_0^1 dx \int_0^{\frac{1}{2}\sqrt{1-x^2}} \frac{(1+x^2)}{\sqrt{1-x^2}\,\sqrt{(1-x^2)-y^2}}\, dy$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\, dx \int_0^{\frac{1}{2}\sqrt{1-x^2}} \frac{dy}{\sqrt{(1-x^2)-y^2}}$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\, dx \left[\sin^{-1} \frac{y}{\sqrt{1-x^2}} \right]_0^{\frac{1}{2}\sqrt{1-x^2}}$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\, dx \left[\frac{\pi}{6} - 0 \right] = \frac{\pi}{6} \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\, dx$$

Let $x = \sin\theta$

$$= \frac{\pi}{6} \int_0^{\pi/2} \frac{(1+\sin^2\theta)}{\cos\theta} \cos\theta\, d\theta$$

$$= \frac{\pi}{6} \left[\frac{\pi}{2} + \frac{1}{2}\frac{\pi}{2} \right] = \frac{\pi}{6}\left(\frac{3\pi}{4}\right) = \frac{\pi^2}{8}$$

Soln. :

(Q32) Let $I = \int_0^a \int_y^{\sqrt{ay}} \frac{x}{x^2 + y^2} dx \cdot dy$

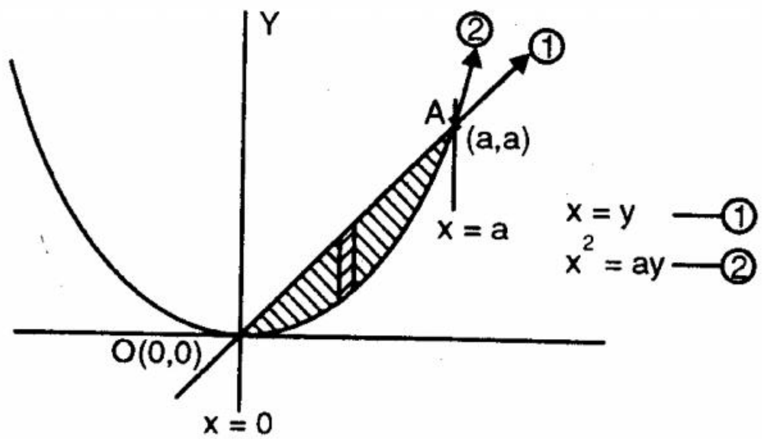


Fig. Ex. 9.35

Given strip is parallel to X-axis.

limits are : $x = y$ to $x = \sqrt{ay}$

$y = 0 \quad \text{to} \quad y = a.$

Now, $x = \sqrt{ay} \Rightarrow x^2 = ay$ is a parabola. The points of intersection of $x = y$ and $x^2 = ay$ are :

$x^2 = ax \qquad \therefore x = 0, a$

$\therefore \quad y = 0, a$

$\therefore$ The points are O (0, 0), A (a, a). We sketch the curve.

We take strip parallel to Y-axis and

$$I = \int_0^a dx \int_{\frac{x^2}{a}}^{x} \frac{x}{x^2+y^2}\,dy = \int_0^a x\,dx \left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_{\frac{x^2}{a}}^{x}$$

$$= \int_0^a x \cdot \frac{1}{x} \cdot \left[\tan^{-1} 1 - \tan^{-1}\frac{x}{a}\right] dx = \int_0^a \left[\frac{\pi}{4} - \tan^{-1}\frac{x}{a}\right] dx = \frac{\pi}{4} \cdot a - \int_0^a \tan^{-1}\frac{x}{a} \cdot dx$$

$$= \frac{\pi a}{4} - \left[\left(x \tan^{-1}\frac{x}{a}\right)_0^a - \int_0^a x \cdot \frac{1}{1+\frac{x^2}{a^2}} \frac{1}{a}\,dx\right] = \frac{\pi a}{4} - \left[(a \tan^{-1} 1) - a\int_0^a \frac{x}{x^2+a^2}\,dx\right]$$

$$= \frac{\pi a}{4} - \left[\frac{\pi a}{4} - \frac{a}{2}\left\{\log(x^2+a^2)\right\}_0^a\right] = \frac{\pi a}{4} - \frac{\pi a}{4} + \frac{a}{2}\log(2a^2) - \frac{a}{2}\log(a^2)$$

$$= \frac{a}{2}\log(2) = a\log(2)^{1/2} = a \log\sqrt{2}$$

Soln. :

(032) Let $I = \int_0^a \int_0^x \frac{dx\,dy}{(y+a)\sqrt{(a-x)(x-y)}}$

We change the order of integration as it is not convenient to integrate the inner integral w.r.t. y.

Limits are : $y = 0$ to $y = x$

and $x = 0$ to $x = a$.

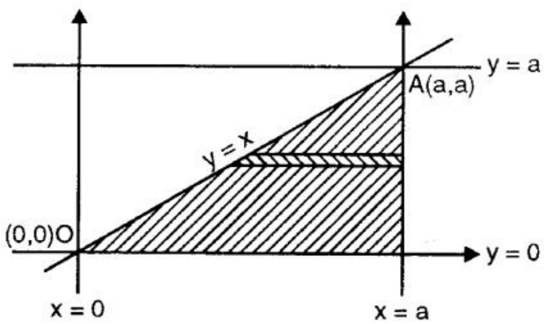


Fig. Ex. 9.39

Taking strip parallel to X-axis.

$$I = \int_0^a \int_y^a \frac{dx\,dy}{(y+a)\sqrt{(a-x)(x-y)}}$$

$$= \int_0^a \frac{dy}{y+a}\left[\int_y^a \frac{dx}{\sqrt{(a-x)(x-y)}}\right]$$

Note : Limits a and b of integration can be changed to 0 and 1 by putting $x - a = (b-a)t$.

Let $x - y = (a-y)t$

$\therefore\ dx = (a-y)\,dt$

x	y	a
t	0	1

$$\therefore\ I = \int_0^a \frac{dy}{y+a}\left[\int_0^1 \frac{(a-y)\,dt}{\sqrt{(a-y)(1-t)(a-y)t}}\right]$$

$(\because x - y = (a-y)(1-t))$

$$= \int_0^a \frac{dy}{y+a}\int_0^1 t^{-1/2}\cdot(1-t)^{-1/2}\cdot dt = [\log(y+a)]_0^a\, B\left(\frac{1}{2},\frac{1}{2}\right)$$

$$= [\log(2a) - \log a]\,\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 1} = (\log 2)\left(\frac{\sqrt{\pi}\,\sqrt{\pi}}{1}\right) = \pi \log 2$$

234) We change the order as it is convenient to integrate the inner integral w.r.t. x.

Limits are :

$$y = 0 \text{ to } y = \frac{1}{x} \text{ (i.e. } xy = 1)$$

$$x = 0 \text{ to } x = 1.$$

Sketch :

We take strip parallel to X-axis. We have to divide the region into two parts, say R_1 and R_2.

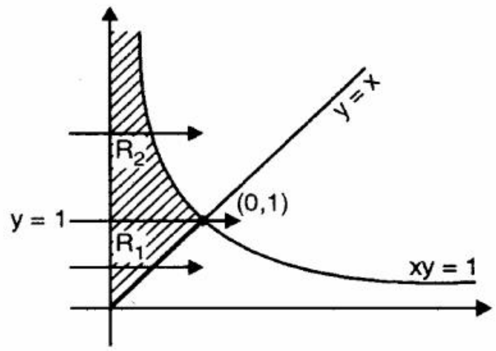


Fig. Ex. 9.42

For R_1 :

$$I_1 = \iint_{R_1} \frac{y}{(1+xy)^2(1+y^2)} dx \cdot dy$$

$$= \int_0^1 \frac{y}{1+y^2} dy \int_0^y (1+xy)^{-2} \cdot dx = \int_0^1 \frac{y}{1+y^2} dy \left[\frac{(1+xy)^{-1}}{-y}\right]_0^y$$

$$= -\int_0^1 \frac{dy}{1+y^2}\left[\frac{1}{1+y^2} - 1\right] = -\int_0^1 \frac{dy}{(1+y^2)^2} + \int_0^1 \frac{dy}{1+y^2}$$

Let $y = \tan\theta$, $dy = \sec^2\theta \cdot d\theta$

$$= -\int_0^{\pi/4} \frac{\sec^2\theta \cdot d\theta}{\sec^4\theta} + [\tan^{-1} y]_0^1 = -\int_0^{\pi/4} \cos^2\theta \cdot d\theta + \left(\frac{\pi}{4} - 0\right)$$

$$= -\frac{1}{2}\int_0^{\pi/4} (1+\cos 2\theta)\, d\theta + \frac{\pi}{4} = -\frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} + \frac{\pi}{4}$$

$$= -\frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\right] + \frac{\pi}{4} = -\frac{\pi}{8} - \frac{1}{4} + \frac{\pi}{4}$$

$$= \frac{\pi}{8} - \frac{1}{4} \quad \text{...(i)}$$

For R_2 :

$$I_2 = \int_1^{\infty}\int_0^{1/y} \frac{y}{(1+xy)^2(1+y^2)} dx\, dy$$

$$= \int_1^\infty \frac{y}{(1+y^2)}\,dy \int_0^{1/y} (1+xy)^{-2} \cdot dx$$

$$= \int_1^\infty \frac{y}{(1+y^2)}\,dy \left[\frac{(1+xy)^{-1}}{-y} \right]_0^{1/y}$$

$$= -\int_1^\infty \frac{dy}{(1+y^2)} \left[\frac{1}{2} - 1 \right]$$

$$= \frac{1}{2} \int_1^\infty \frac{dy}{(1+y^2)} = \frac{1}{2} [\tan^{-1} y]_1^\infty$$

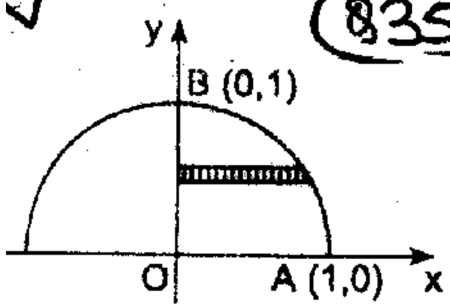
$$= \frac{1}{2} \left[\frac{\pi}{2} - \frac{\pi}{4} \right] = \frac{\pi}{8} \qquad \text{...(ii)}$$

From (i) and (ii)

$$I = I_1 + I_2$$

$$= \frac{\pi}{8} - \frac{1}{4} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{4} = \frac{1}{4}(\pi - 1)$$

(835)



Sol. : The limits of y are 0 and $\sqrt{1-x^2}$ and those of x are 0 and 1. We therefore, draw the curve $y = \sqrt{1-x^2}$ *i.e.* the upper-half of the circle $x^2 + y^2 = 1$. The region of integration is OAB.

To change the order of integration we consider a strip parallel to x-axis. Now, x varies from 0 to $\sqrt{1-y^2}$ and y varies from 0 to 1.

$$\therefore \quad I = \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{e^y}{(e^y+1)\sqrt{(1-y^2)-x^2}}\, dy\, dx$$

$$= \int_0^1 \frac{e^y}{(e^y+1)} \left[\sin^{-1} \frac{x}{\sqrt{1-y^2}}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \frac{e^y}{e^y+1} \cdot \frac{\pi}{2} \cdot dy = \frac{\pi}{2}\left[\log(e^y+1)\right]_0^1$$

$$= \frac{\pi}{2} \log\left(\frac{e+1}{2}\right).$$

(Q38) Change the order of integration in $\int_{0} \int_{2-x} f(x, y)\, dx\, dy$ **(Dec. 96, 6 Marks)**

Sol. :

(1) The limits are :

$y = 2 - x$ to $y = 2 + x$

and $x = 0$ to $x = 5$

(2) We trace all the four curves :

(3) Given strip is parallel to Y- axis with its lower end on the line $y = 2 - x$ and upper end on $y = 2 + x$.

(4) **This strip is moved from $x = 0$ to $x = 5$. and the Δ PQR is the region of integration.**

(5) We change the order of integration i.e. **we take strip** parallel to X -axis.

(6) We observe that the horizontal strip can rest on the two different lines on its left.

So we divide the region into two sub-regions R_1 and R_2 by drawing a line PS parallel to the strip.

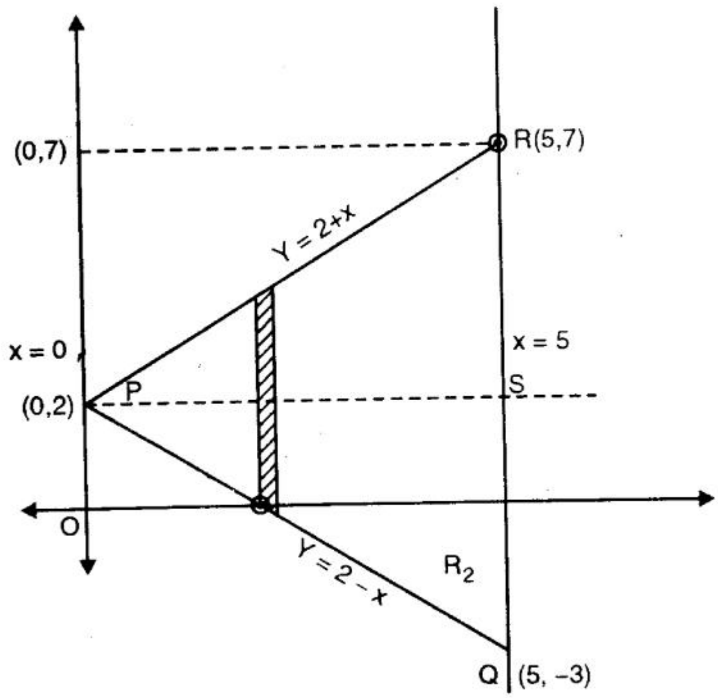


Fig. 9.10

Now in the region R_1, left end of the strip is on $x = y - 2$ and right end on $x = 5$ and strip can move from $y = 2$ to $y = 7$.

And in the region R_2 , left end of the strip is on $x = 2 - y$ and right end on $x = 5$ and strip can move from $y = -3$ to $y = 2$.

$\therefore$ R_1 and R_2 together give the region

$$\therefore \quad I = \int_{2}^{7} \int_{y-2}^{5} f(x, y)\, dx\, dy + \int_{-3}^{2} \int_{2-y}^{5} f(x, y)\, dx\, dy$$

Q39 The limits are

$$x = y \quad \text{to} \quad x = 2 + \sqrt{4 - 2y}$$

and $\quad y = 0 \quad$ to $\quad y = 2$

The strip is parallel to X-axis.

Also, $\quad x = 2 + \sqrt{4-2y}$

$$\Rightarrow \quad (x-2)^2 = 4 - 2y = -2(y-2)$$

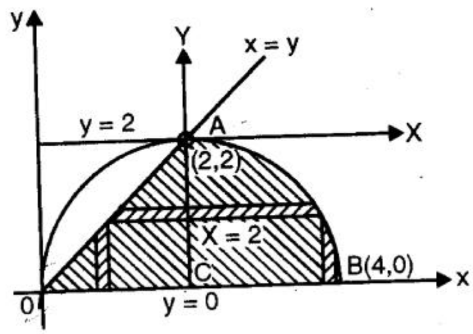


Fig. Ex. 9.1

The equation represents parabola, convex upwards with vertex at (2, 2) and it passes through origin.

Region of integration is as shaded-one.

We choose strip parallel to Y-axis. Since the upper end of the strip changes its curve at A, we break the integral into two integrals.

We write,

$$I = \int_0^2 \int_0^x f(x, y)\, dx\, dy + \int_2^4 \int_0^{\frac{4x - x^2}{2}} f(x, y)\, dx\, dy$$

Soln. :

(Q40)

Let $I = \int_{0}^{2} \int_{\sqrt{4-x}}^{(4-x)^2} f(x, y)\, dx\, dy$

Given limits are :

$$y = \sqrt{4-x} \text{ to } y = (4-x)^2$$

and $x = 0$ to $x = 2$.

Now, $y = \sqrt{4-x}$

$\therefore\ y^2 = -(x-4)$ is a parabola with vertex at (4, 0) and convex to right.

Also, $y = (4-x)^2$ is a parabola with vertex at (4, 0) and convex downwards.

The region of integration is as shaded.

The parabolas intersect at (3, 1).

The line $x = 2$ divides the region into three parts :

They are given by $y = 2$, $y = 4$. Let the regions be R_1, R_2, R_3.

where, $R_1 \to ABC$, $R_2 \to BCDE$, $R_3 \to DEF$

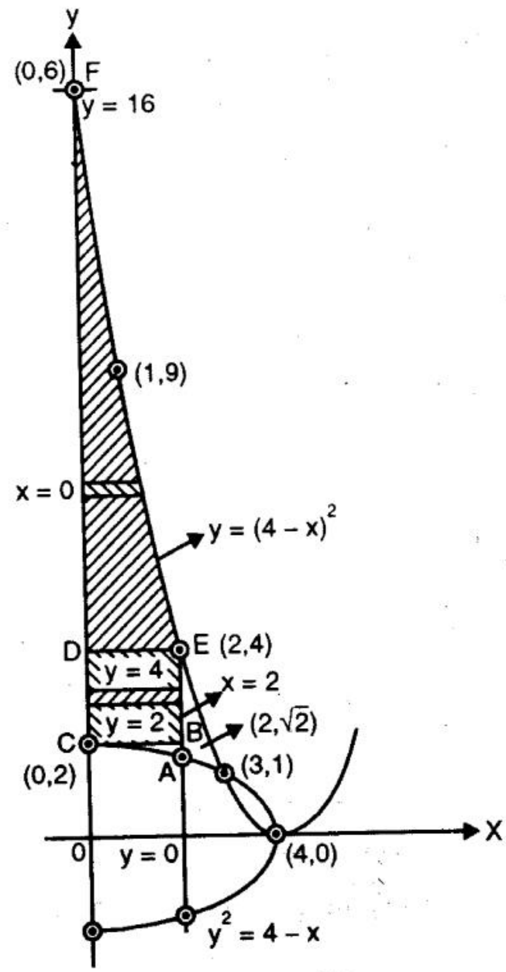


Fig. Ex. 9.4

To change the order, we take strip parallel to X-axis and

$$I = \int_{\sqrt{2}}^{2} \int_{4-y^2}^{2} f(x, y)\, dx\, dy + \int_{2}^{4} \int_{0}^{2} f(x, y)\, dx\, dy$$

$$+\int_{4}^{16}\int_{0}^{4-\sqrt{y}} f(x, y)\,dx \cdot dy$$

Soln. : Given limits are :

Q41

$$y = x \quad \text{to} \quad y = 1 + \sqrt{1 - x^2}$$

and $x = 0$ to $x = 1$

Given strip is parallel to Y-axis.

Also, $y = 1 + \sqrt{1 - x^2} \Rightarrow$

$$(y - 1)^2 = 1 - x^2$$

i.e. $x^2 + (y - 1)^2 = 1$ is a circle; centre at (0, 1) and radius 1.

On simplification, $x^2 + y^2 = 2y$

$$\therefore \quad x = \pm\sqrt{2y - y^2}$$

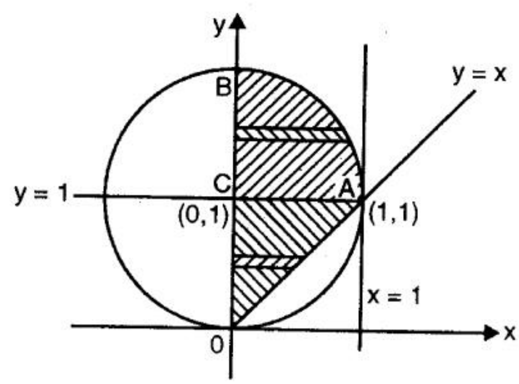


Fig. Ex. 9.9

Shaded part is region of integration.

We choose the strip parallel to X-axis. Right end of the strip changes its curve at y = 1, so we have to write two integrals.

$$I = \iint_{R_1} f(x, y)\,dx\,dy + \iint_{R_2} f(x, y)\,dx \cdot dy$$

where $R_1 = OAC_1$

$R_2 = ABC$

$$= \int_0^1 \int_0^y f(x, y)\,dx\,dy + \int_1^2 \int_0^{\sqrt{2y - y^2}} f(x, y)\,dx\,dy$$

(Q42) Limits :-

$x = \frac{y}{2}$ to $x = 9 - y$ and

$y = 0$ to $y = 4$

The point of intersection of $x = \frac{y}{2}$ and $x = 9 - y$ is (3, 6).

Given strip is parallel to X-axis.

The region is OABCDEO which is shaded.

We choose the strip parallel to Y-axis. It divides the region into three parts, say, R_1, R_2 and R_3

where $R_1 \to OAE$,

$R_2 \to ABDE$,

$R_3 \to BCD$.

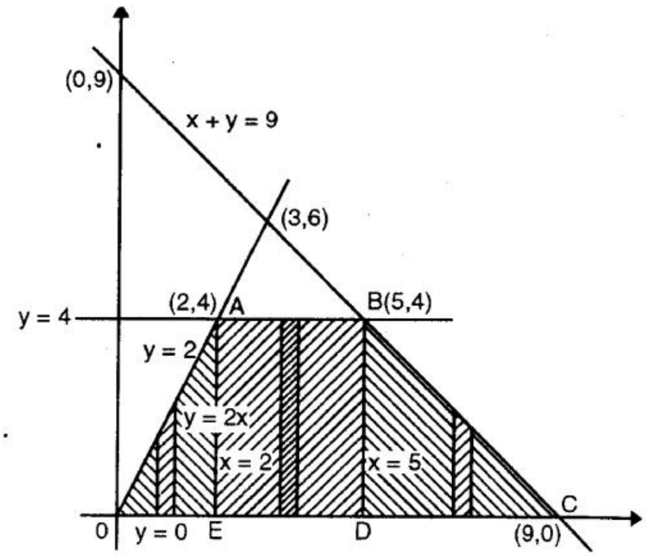


Fig. Ex. 9.10

$$\therefore \quad I = \int_0^2 \int_0^{2x} f(x, y)\, dx\, dy + \int_2^5 \int_0^4 f(x, y)\, dx\, dy$$

$$+ \int_5^9 \int_0^{9-x} f(x, y)\, dx \cdot dy$$

Soln. :
(043)

Let
$$I = \int_0^a \int_{\frac{1}{2}\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y)\, dx \cdot dy$$

Limits are : $y = \frac{1}{2}\sqrt{a^2-x^2}$ to $y = \sqrt{a^2-x^2}$

and $x = 0$ to $x = a$

Now, $y = \frac{1}{2}\sqrt{a^2-x^2} \Rightarrow 4y^2 + x^2 = a^2$

i.e. $\frac{x^2}{4} + \frac{y^2}{1} = \frac{a^2}{4}$

i.e. $\frac{x^2}{a^2} + \frac{y^2}{\left(\frac{a}{2}\right)^2} = 1$ is an ellipse.

and $y = \sqrt{a^2-x^2} \Rightarrow x^2 + y^2 = a^2$ is a circle.

The points of intersection are $(-a, 0)$ and $(a, 0)$

The sketch :

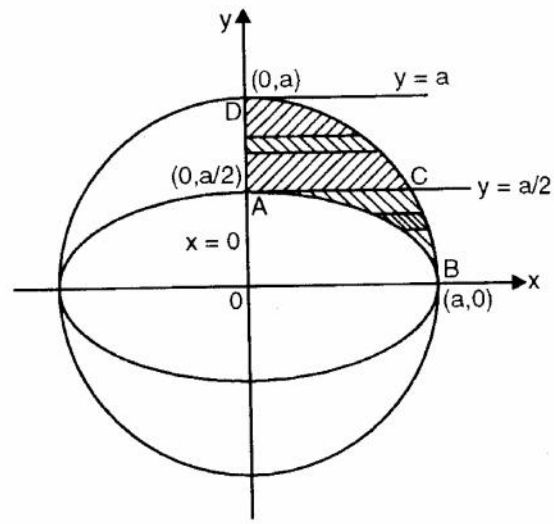


Fig. Ex. 9.16

We have , $x^2 + 4y^2 = a^2$

$\therefore \quad x^2 = a^2 - 4y^2$

$\therefore \quad x = \pm\sqrt{a^2 - 4y^2}$

We choose $x = +\sqrt{a^2-4y^2}$; since it is in first quadrant.

And also, $x^2 + y^2 = a^2$

$\therefore \quad x = \pm\sqrt{a^2-y^2}$

We choose $x = \sqrt{a^2-y^2}$

Since the given strip is parallel to Y-axis; we take it parallel to X-axis. It changes its left end at $x = \frac{a}{2}$. So we divide the region into two parts. And write

$$I = \int_0^{a/2} \int_{\sqrt{a^2-4y^2}}^{\sqrt{a^2-y^2}} f(x, y)\, dx\, dy + \int_{a/2}^{a} \int_0^{\sqrt{a^2-y^2}} f(x, y)\, dx\, dy$$

Soln. : (Q44)

Let $$I = \int_0^1 \int_{2y}^{2(1+\sqrt{1-y})} f(x, y)\, dx \cdot dy$$

Limits are :

$x = 2y$ to $x = 2(1 + \sqrt{1-y})$

and $y = 0$ to $y = 1$

Strip is parallel to X-axis,

Now, $x = 2y$ is a line and

$$x = 2(1 + \sqrt{1-y})$$

$$\therefore \left(\frac{x}{2} - 1\right)^2 = -(y-1)$$

i.e. $(x-2)^2 = -4(y-1)$ is a parabola with vertex at (2, 1) and is convex upwards.

Also the points of intersection of $x = 2y$ and $x = 2(1 + \sqrt{1-y})$ are given by

$$2y = 2(1 + \sqrt{1-y})$$

$$\therefore (y-1)^2 = 1 - y$$

$$\therefore (y-1)^2 + (y-1) = 0 \qquad \therefore (y-1)[y-1+1] = 0$$

$$\therefore y = 0, 1 \qquad \text{and} \qquad x = 0, 2$$

∴ The points are O (0, 0), A (2, 1).

Also, $$x = 2(1+\sqrt{1-y})$$

$$\therefore \left(\frac{x}{2}-1\right)^2 = 1-y$$

$$\therefore y = 1-\left(\frac{x}{2}-1\right)^2$$

$$= 1-\left(\frac{x^2}{4}-x+1\right)$$

$$= x-\frac{x^2}{4}$$

$$\therefore y = \frac{4x-x^2}{4}$$

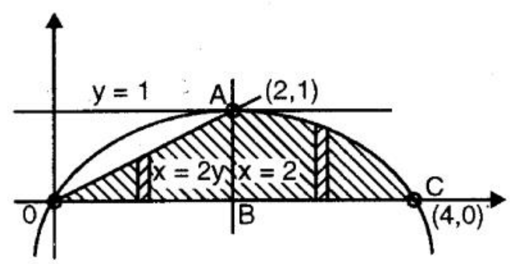


Fig. Ex. 9.26

We choose the strip parallel to Y-axis. And at point, the upper end of the strip changes.

$$\therefore I = \iint_{OAB} f(x,y)\,dx\,dy + \iint_{ABC} f(x,y)\,dx\,dy$$

$$= \int_0^2 \int_0^{x/2} f(x,y)\,dx\cdot dy + \int_2^4 \int_0^{\frac{4x-x^2}{4}} f(x,y)\,dx\cdot dy$$

Soln. :

Q45)

Let, $I = \int_{0}^{2} \int_{\frac{x^2+4}{4}}^{\frac{6-x}{2}} f(x, y)\, dy\, dx$

Given strip is parallel to Y-axis. Limits are $y = \frac{x^2+4}{4}$ to $y = \frac{6-x}{2}$ and $x = 0$ to $x = 2$

$$\text{Now,} \quad y = \frac{x^2+4}{4}$$

$$\therefore \quad x^2 = 4y - 4 = 4(y-1)$$

$\therefore$ $x^2 = 4(y-1)$ is a parabola with vertex at (0, 1) and is convex downwards.

$$\text{Also,} \quad y = \frac{6-x}{2}$$

$\therefore \quad x + 2y = 6$ is a line

To find points of intersection of

$$y = \frac{x^2+4}{4} \text{ and } y = \frac{6-x}{2}$$

$$\therefore \quad \frac{x^2+4}{4} = \frac{6-x}{2}$$

$$\therefore \quad x^2 + 4 = 12 - 2x$$

$$\therefore \quad x^2 + 2x - 8 = 0$$

$$\therefore \quad x = -4, 2$$

$$\therefore \quad y = 5, 2$$

$\therefore$ The points are $(-4, 5)$, $(2, 2)$

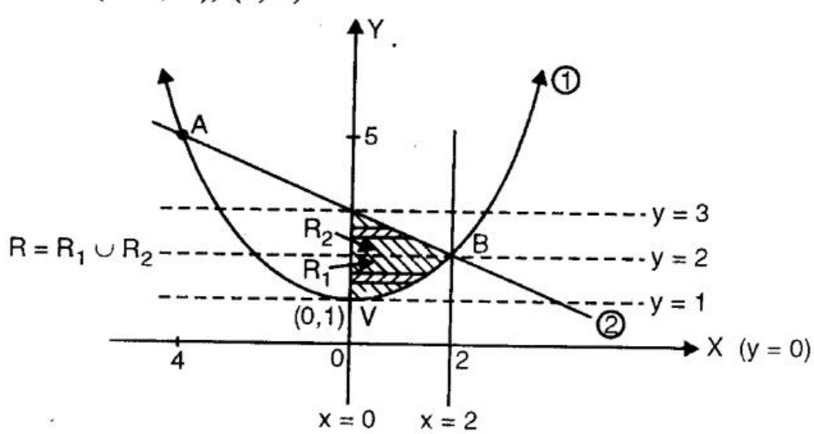


Fig. Ex. 9.31

To change the order, we take strip parallel to X-axis and it changes its right end at $y = 2$. So we divide the region into R_1 and R_2.

For R_1 : limits are, $\quad x = 0$ to $x = 2\sqrt{y-1}$

and $\quad y = 1$ to $y = 2$.

For R_2 : limits are, $\quad x = 0$ to $x = 6 - 2y$

$y = 2$ to $y = 3$

$$\therefore \quad I = \int_1^2 \int_0^{2\sqrt{y-1}} f(x, y)\, dx\, dy + \int_2^3 \int_0^{6-2y} f(x, y)\, dx\, dy.$$

Soln. :

Q46)

Let $I = \int_0^1 \int_{\sqrt{2x-x^2}}^{1+\sqrt{1-x^2}} f(x,y)\,dx \cdot dy$

Given strip is parallel to Y-axis.

limits are : $y = \sqrt{2x - x^2}$ to $y = 1 + \sqrt{1-x^2}$

and $x = 0$ to $x = 1$.

Now, $y = \sqrt{2x - x^2}$

$\therefore \quad x^2 + y^2 - 2x = 0$

is a circle with centre at (1, 0), and radius 1.

Now, $y = 1 + \sqrt{1 - x^2}$

$(y-1)^2 + x^2 = 1$ is a circle with centre at (0, 1) and radius 1.

To find the points of intersection of $x^2 + y^2 = 2x$ and $x^2 + y^2 - 2y = 0$

$\therefore \quad x^2 + y^2 = 2x = 2y$

$\therefore \quad x = y$

$\therefore \quad x^2 + x^2 = 2x$

$\therefore \quad x^2 = x$

$\therefore \quad x = 0, 1$

$y = 0, 1$

$\therefore$ The points are O (0, 0), A(1, 1).

We choose the strip parallel to X-axis. It changes its right end at y = 1. So, we divide the region into two parts, say R_1 and R_2.

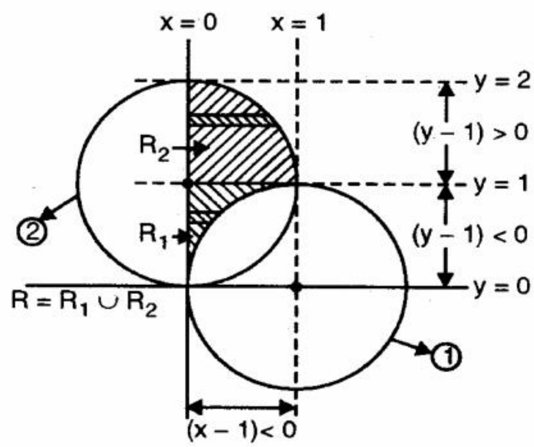


Fig. Ex. 9.32

Limits for R_1 : $x = 0$ to $x = 1 - \sqrt{1-y^2}$

$y = 0$ to $y = 1$

and limits for R_2 : $x = 0$ to $x = \sqrt{2y - y^2}$

$y = 1$ to $y = 2$.

We write,

$$I = \iint_{R_1} + \iint_{R_2}$$

$$\therefore \quad I = \int_0^1 \int_0^{1-\sqrt{1-y^2}} f(x, y)\, dx\, dy + \int_1^2 \int_0^{\sqrt{2y-y^2}} f(x, y)\, dx\, dy$$

Soln. :

(Q47)

$$\text{Let} \quad I = \int_0^8 \int_{\frac{y-8}{4}}^{y/4} f(x, y)\, dx \cdot dy$$

Given strip is parallel to X-axis limits are :

$$x = \frac{y-8}{4} \text{ to } x = \frac{y}{4}$$

$$\text{and} \quad y = 0 \text{ to } y = 8$$

$$\text{Now,} \quad x = \frac{y-8}{4} \Rightarrow 4x + 8 = y$$

$$\text{and} \quad x = \frac{y}{4} \Rightarrow y = 4x \text{ are straight lines.}$$

The lines are parallel, since their slopes are same.

All the curves are lines. We draw the sketch.

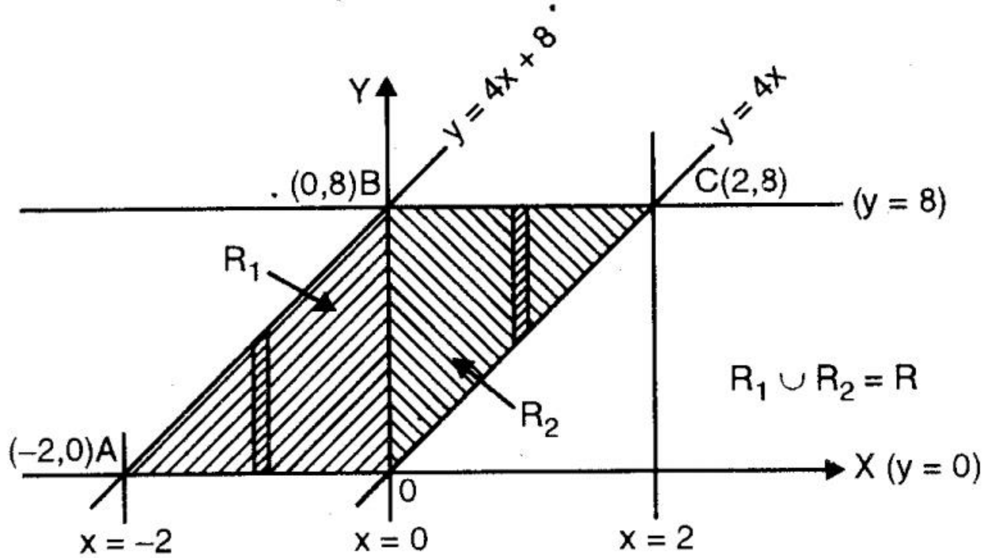


Fig. Ex. 9.36

To change the order, we take strip parallel to Y-axis.

We divide the region into two parts R_1 and R_2.

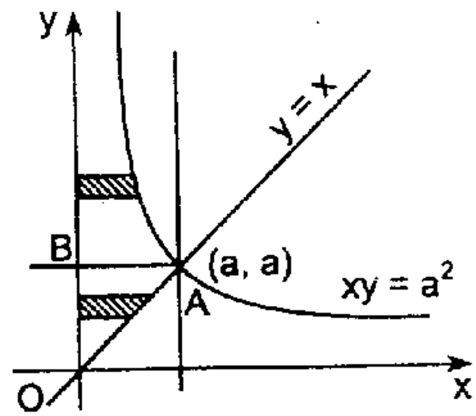
We write,

$$I = \int_{-2}^{0} \int_{y=0}^{4x+8} f(x, y)\, dx\, dy + \int_{0}^{2} \int_{4x}^{8} f(x, y)\, dx\, dy$$

$\int_0 \int_x f(x, y)\, dx\, dy.$

(848)

Sol. : The region of integration is given by $y = x$, a line through the origin, $y = a^2 / x$ i.e. $xy = a^2$, a rectangular hyperbola, $x = 0$, the y-axis and $x = a$, a line parallel to the y-axis.



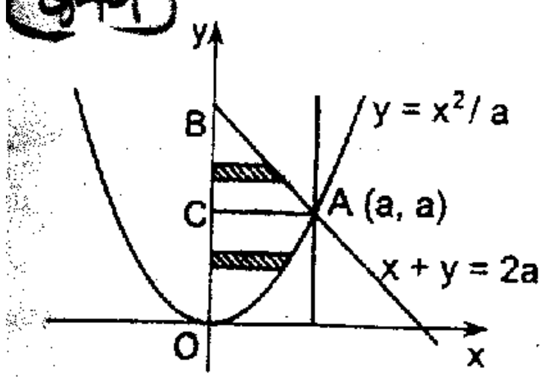
If we have to change the order of integration, the region is to be divided into two parts OAB and above the line AB.

In the region OAB x varies from 0 to $x = y$ and y varies from 0 to $y = a$. In the second region x varies from $x = 0$ to $x = a^2 / y$ and y varies from a to ∞.

$$\therefore \quad I = \int_0^a \int_0^y f(x, y)\, dy\, dx + \int_a^\infty \int_0^{a^2/y} f(x, y)\, dx\, dy.$$

Sol. : The region of integration is given by $y = x^2 / a$ i.e. a parabola, $y = 2a - x$ i.e. $x + y = 2a$ i.e. a line, $x = 0$ i.e. the y-axis and $x = a$ i.e. a line parallel to the y-axis.

(849)



If we have to change the order of integration, the region is divided into two parts OAC and CAB.

In the region OAC, x varies from 0 to $\sqrt{ay}$ and y varies from 0 to a [The point of intersection A of the parabola $y = x^2 / a$ and the line $x + y = 2a$ is $A\,(a, a)$.] In the region CAB, x varies from 0 to $2a - y$ and y varies from a to $2a$. Hence,

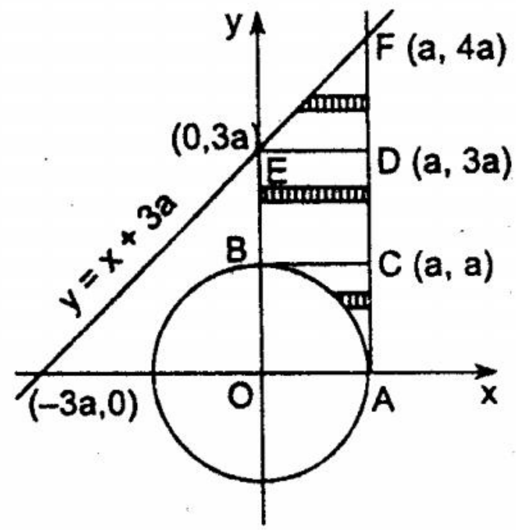
$$\therefore \quad I = \int_0^a \int_0^{\sqrt{ay}} f(x, y)\, dx\, dy + \int_a^{2a} \int_0^{2a-y} f(x, y)\, dx\, dy.$$

(850) $\int_0 \int_{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy.$ **(M.U. 1990)**

Sol. : The region of integration is given by $y=\sqrt{a^2-x^2}$ i.e. $x^2+y^2=a^2$, a circle, $y=x+3a$, a straight line, $x=0$, the y axis and $x=a$, a line parallel to the y-axis.

A-10

If we have to change the order of integration the region is divided into three parts ACB, $CDEB$ and DEF.



In the region ACB, x varies from $\sqrt{a^2-y^2}$ to a and y varies from 0 to a.

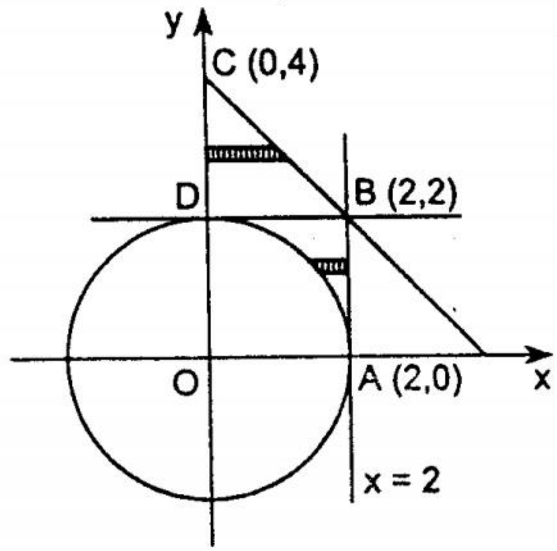
In the region $CDEB$, x varies from 0 to a and y varies from a to $3a$.

In the region DEF, x varies from $y-3a$ to a and y varies from $3a$ to $4a$. Hence,

$$\therefore\quad I=\int_0^a\int_{\sqrt{a^2-y^2}}^a f(x,y)\,dx\,dy+\int_a^{3a}\int_0^a f(x,y)\,dx\,dy+\int_{3a}^{4a}\int_{y-3a}^a f(x,y)\,dx\,dy$$

(Q51)

Sol. : The region of integration is given by $y = \sqrt{4 - x^2}$ i.e. $x^2 + y^2 = 2^2$, a circle, $y = 4 - x$ i.e. $x + y = 4$, a straight line, $x = 0$, the y-axis and $x = 2$ the line parallel to the y-axis. Thus, the region of integration is $ABCD$.



When we change the order of integration, the region is split into two parts DAB and DBC.

In the region DAB, x varies from $x = \sqrt{4 - y^2}$ to $x = 2$ and y varies from 0 to 2.

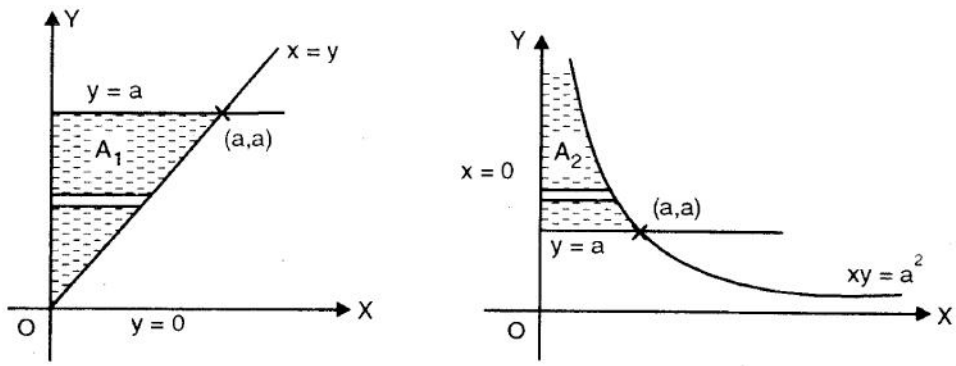
In the region DBC, x varies from $x = 0$ to $x = 4 - y$ and y varies from $y = 2$ to $y = 4$.

$$\therefore \quad I = \int_0^2 \int_{\sqrt{4-y^2}}^2 f(x, y)\, dx\, dy + \int_2^4 \int_0^{\sqrt{4-y}} f(x, y)\, dx\, dy.$$

Q55) **Sol. :**

Note : Sum of two integrals can be reduced to a single integral by changing the order of integration.

Region of integration for first integral on R.H.S. is bounded by the lines $x = 0$, $x = y$, $y = 0$, $y = a$ it is region A_1 shown shaded in Fig. 9.11.



(a) **(b)**

Fig. 9.11

Region for second integral on R.H.S. is bounded by $x = 0$, $x = \frac{a^2}{y}$ i.e. $xy = a^2$

$y = a$ and $y = \infty$. It is region A_2. The region A_2 extends up to infinity in Y - direction. We superimpose the two regions.

We combine the two regions and get one region.

We change the order of integration i.e. takes strip parallel to Y- axis. $x = 0$ in the combined region.

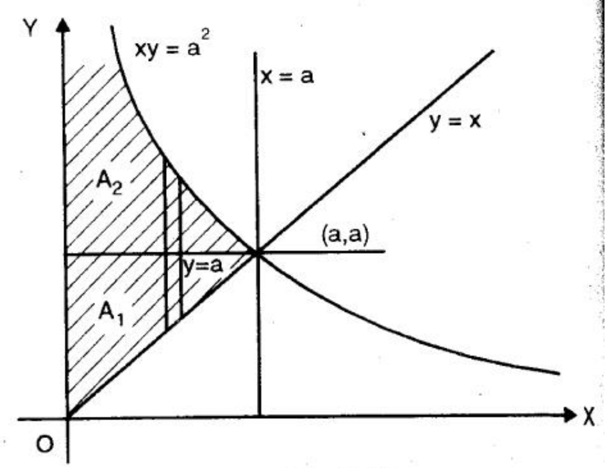


Fig. 9.12

The lower end of the strip is on $y = x$ and upper end on $y = \frac{a^2}{x}$ and to covers the whole region we move the strip parallel to itself from $x = 0$ to $x = a$.

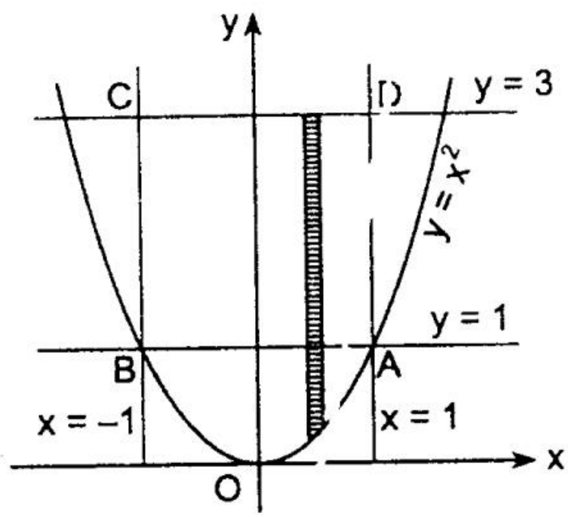
$$\text{Hence} \quad I = \int_0^a \int_x^{a^2/x} f(x, y)\, dx\, dy$$

Q56 $I = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dy\,dx + \int_1^3 dy \int_{-1}^{1} dx$. **(M.U. 2000)**

Sol. : Let $I = I_1 + I_2$.

Now, for I_1, the limits are $x = -\sqrt{y}$ and $x = \sqrt{y}$ i.e. $x^2 = y$, a parabola with vertex at the origin and opening upwards. The limits for y are $y = 0$ to $y = 1$. **The region is *OAB*.**

For I_2, x varies from $x = -1$ to $x = 1$ and y varies from $y = 1$ to $y = 3$. **The region is *ABCD*.** We have, to sweep both the regions *i.e.* the region *OABDCBO*.



Now, consider a strip parallel to the x-axis extending from the parabola to the line CD.

On this strip y varies from $y = x^2$ to $y = 3$. To sweep the whole area the strip has to move from $x = -1$ to $x = 1$.

$$\therefore \quad I = \int_{-1}^{1} \int_{x^2}^{3} dy\,dx = \int_{-1}^{1} [\,y\,]_{x^2}^{3} dx$$

$$= \int_{-1}^{1} \left[3 - x^2\right] dx = 2\int_0^1 (3 - x^2)\,dx \qquad [\because \text{Even function}]$$

$$= 2\left[3x - \frac{x^3}{3}\right]_0^1 = 2\left[3 - \frac{1}{3}\right] = \frac{16}{3}.$$

Soln. :
Q57)

Let $I = \iint_R r\sqrt{a^2 - r^2}\, dr\, d\theta$

$$\therefore \quad I = \int_0^{\pi/2} d\theta \int_0^{a\cos\theta} r\sqrt{a^2 - r^2}\, dr$$

$$= -\frac{1}{2}\int_0^{\pi/2} d\theta \int_0^{a\cos\theta} (-2r)\sqrt{a^2 - r^2}\, dr$$

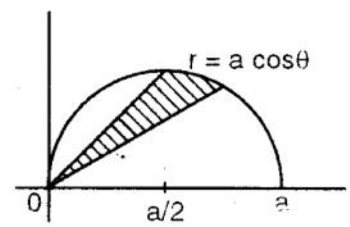


Fig. Ex. 9.7

$$= -\frac{1}{2}\int_0^{\pi/2} \left[\frac{2}{3}(a^2 - r^2)^{3/2}\right]_0^{a\cos\theta} \cdot d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2} \left[a^3\sin^3\theta - a^3\right] d\theta = -\frac{a^3}{3}\int_0^{\pi/2} \sin^3\theta \cdot d\theta + \frac{a^3}{3}\int_0^{\pi/2} d\theta$$

$$= -\frac{a^3}{3}\cdot\frac{2}{3}\cdot 1 + \frac{a^3}{3}\cdot\frac{\pi}{2} = \frac{a^3}{18}(3\pi - 4)$$

Soln. :

258)

Let $$I = \int_0^{4a} \int_{\frac{y^2}{4a}}^{y} \frac{x^2 - y^2}{x^2 + y^2}\, dx \cdot dy$$

limits are :

$$x = \frac{y^2}{4a} \text{ to } x = y$$

and $y = 0$ to $y = 4a$

Now, $y^2 = 4ax$ is a parabola and the point of intersection of $y^2 = 4ax$ and $x = y$ is $(0, 0)$, $(4a, 4a)$.

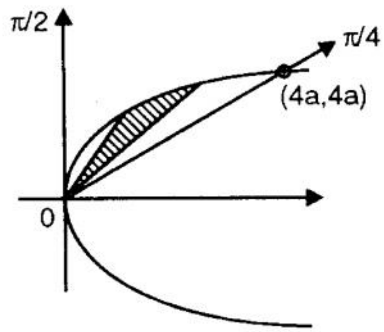


Fig. Ex. 9.15

Changing to polar;

i.e. $x = r\cos\theta$, $y = r\sin\theta$;

the equation $y^2 = 4ax$ becomes

$$r^2 \sin^2\theta = 4ar\cos\theta$$

$$\therefore \quad r = \frac{4a\cos\theta}{\sin^2\theta}$$

and $x = y \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore \quad I = \int_{\pi/4}^{\pi/2} \int_{0}^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2}\, r\, dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} d\theta \int_{0}^{\frac{4a\cos\theta}{\sin^2\theta}} (\cos^2\theta - \sin^2\theta)\, r\, dr$$

$$= \int_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\left[\frac{r^2}{2}\right]_0^{\frac{4a\cos\theta}{\sin^2\theta}}$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\left[\frac{16a^2\cos^2\theta}{\sin^4\theta} - 0\right]\cdot d\theta$$

$$= \frac{16a^2}{2}\int_{\pi/4}^{\pi/2}\left[\frac{\cos^4\theta}{\sin^4\theta} - \frac{\cos^2\theta}{\sin^2\theta}\right] d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} (\cot^4\theta - \cot^2\theta)\, d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} \left[\cot^2\theta(\operatorname{cosec}^2\theta - 1) - (\operatorname{cosec}^2\theta - 1)\right] d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} (\cot^2\theta - 1)(\operatorname{cosec}^2\theta - 1)\, d\theta$$

$$= 8a^2 \int^{\pi/2} (\cot^2\theta - 1)\operatorname{cosec}^2\theta\, d\theta - 8a^2 \int^{\pi/2} (\operatorname{cosec}^2\theta - 2)\, d\theta$$

$$= 8a^2\left[-\frac{\cot^3\theta}{3}-\theta\right]_{\pi/4}^{\pi/2} - 8a^2\left[-\cot\theta-2\theta\right]_{\pi/4}^{\pi/2}$$

$$= 8a^2\left[-\frac{\pi}{2}+\frac{1}{3}+\frac{\pi}{4}\right]-8a^2\left[-\pi+1+\frac{\pi}{2}\right]$$

$$= 8a^2\left[-\frac{\pi}{2}+\frac{1}{3}+\frac{\pi}{4}+\pi-1-\frac{\pi}{2}\right]=8a^2\left(\frac{\pi}{4}-\frac{2}{3}\right)$$

Soln. :

859)

Let $$I = \iint\limits_R x^{m-1} \cdot y^{n-1} \cdot dx \cdot dy$$

We transform it to elliptical polar.

Let $x = ar\cos\theta$, $y = br\sin\theta$, $dxdy = abr\,dr\,d\theta$

$$\therefore \quad I = \int_0^{\pi/2} \int_0^1 (ar\cos\theta)^{m-1} \cdot (br\sin\theta)^{n-1} \cdot abr\,dr\,d\theta$$

$$= a^m \cdot b^n \int_0^{\pi/2} (\sin\theta)^{n-1} \cdot (\cos\theta)^{m-1} \cdot d\theta \int_0^1 r^{m+n-1} \cdot dr$$

$$= a^m b^n \cdot \frac{1}{2} B\left(\frac{n}{2}, \frac{m}{2}\right) \cdot \left[\frac{r^{m+n}}{m+n}\right]_0^1$$

$$= \frac{1}{2} a^m b^n \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+n}{2}\right)} \cdot \frac{1}{m+n} = \frac{1}{2} a^m b^n \left[\frac{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}{\left(\frac{m+n}{2}\right)\Gamma\left(\frac{m+n}{2}\right)}\right] \cdot \frac{2}{n} \cdot \frac{1}{2}$$

$$= \frac{1}{2n} a^m b^n \frac{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+n}{2}+1\right)}$$

Soln. :

Q60)

Let $I = \int_0^1 \int_0^x (x+y)\, dx \cdot dy$

To find region of integration :

limits are : $y = 0$ to $y = x$

$x = 0$ to $x = 1$

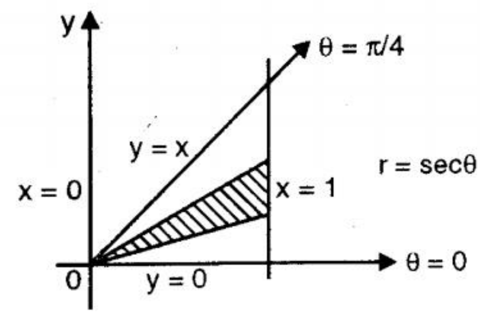


Fig. Ex. 9.22

Changing to polar,

$$y = x \Rightarrow r \sin\theta = r\cos\theta \Rightarrow \theta = \frac{\pi}{4}$$

$$x = 1 \Rightarrow r\cos\theta = 1 \Rightarrow r = \sec\theta$$

$$\therefore \quad I = \int_0^{\pi/4} \int_0^{\sec\theta} r(\cos\theta + \sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/4} (\cos\theta + \sin\theta)\, d\theta \left(\frac{r^3}{3}\right)_0^{\sec\theta}$$

$$= \frac{1}{3}\int_0^{\pi/4} (\cos\theta + \sin\theta)\sec^3\theta \cdot d\theta$$

$$= \frac{1}{3}\int_0^{\pi/4} \left[\sec^2\theta + \tan\theta\sec^2\theta\right] d\theta$$

$$= \frac{1}{3}\left[\tan\theta + \frac{\tan^2\theta}{2}\right]_0^{\pi/4} = \frac{1}{3}\left[1 + \frac{1}{2}\right] = \frac{1}{2}$$

Soln. :

961)

Let $I = \iint_R \sqrt{\dfrac{1-x^2-y^2}{1+x^2+y^2}}\, dx \cdot dy$

changing to polar :

Let $x = r\cos\theta,\ \ y = r\sin\theta,$

$dxdy = r\,dr\,d\theta$

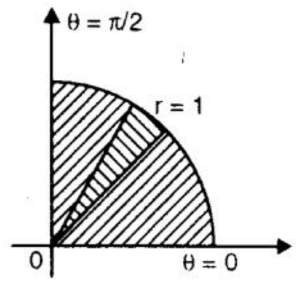


Fig. Ex. 9.28

$$\therefore \quad I = \int_0^{\pi/2}\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}} \cdot r\,dr\,d\theta$$

$$\therefore \quad I = \int_0^{\pi/2} d\theta \int_0^1 \frac{1-r^2}{\sqrt{1-r^4}}\, r\,dr$$

Let $r^2 = \sin\phi$

$\therefore \quad 2r\,dr = \cos\phi\,d\phi$

r	0	1
ϕ	0	$\pi/2$

$$\therefore \quad I = \frac{\pi}{2} \cdot \int_0^{\pi/2} \frac{1-\sin\phi}{\cos\phi} \cdot \frac{\cos\phi}{2} \cdot d\phi$$

$$= \frac{\pi}{4}\left[\frac{\pi}{2} - 1\right]$$

Soln. :

962)

Let $I = \int_0^1 \int_x^{\sqrt{2x - x^2}} (x^2 + y^2)\, dx \cdot dy$

Given strip is parallel to Y-axis : limits are $y = x$ to $y = \sqrt{2x - x^2}$
and $x = 0$ to $x = 1$.

$$\text{Now, } \quad y = \sqrt{2x - x^2}$$

$$\therefore \quad x^2 + y^2 - 2x = 0$$

$$\therefore \quad (x - 1)^2 + y^2 = 1$$

is a circle with centre (1, 0), radius 1.

To find points of intersection of $y = x$ and $y = \sqrt{2x - x^2}$

$$\therefore \quad x = \sqrt{2x - x^2}$$

$$\therefore \quad x^2 = 2x - x^2$$

$$\therefore \quad 2x^2 = 2x$$

$$\therefore \quad x = 0, 1$$

$$y = 0, 1$$

$\therefore$ The points are (0, 0), (1, 1)

The sketch is,

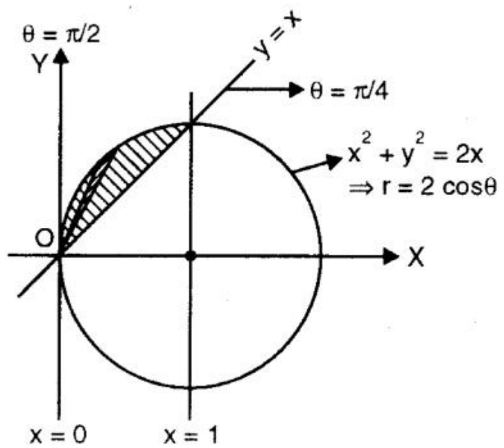


Fig. Ex. 9.33

Changing to polar, equation of circle is $r = 2\cos\theta$ and the line $y = x$ is $\theta = \frac{\pi}{4}$

$$\therefore \quad I = \int_{\pi/4}^{\pi/2} \int_{r=0}^{2\cos\theta} r^2 (r\,dr\,d\theta) = \int_{\pi/4}^{\pi/2} \left(\frac{r^4}{4}\right)_0^{2\cos\theta} \cdot d\theta$$

$$= \frac{1}{4} \int_{\pi/4}^{\pi/2} 16 \cos^4 \theta \cdot d\theta$$

$$= 4 \int_{\pi/4}^{\pi/2} \cos^4 \theta \cdot d\theta \qquad \text{...(i)}$$

Now, $\cos^4 \theta = \left(\frac{1 + \cos 2\theta}{2} \right)^2 = \frac{1}{4}\left[1 + 2 \cos 2\theta + \frac{1 + \cos 4\theta}{2} \right]$

$$= \frac{1}{8} [3 + 4 \cos 2\theta + \cos 4\theta]$$

from (i),

$$I = \frac{4}{8} \int_{\pi/4}^{\pi/2} [3 + 4 \cos 2\theta + \cos 4\theta] \, d\theta = \frac{1}{2}\left[3\theta + \frac{4 \sin 2\theta}{2} + \frac{\sin 4\theta}{4} \right]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{3\pi}{2} + 0 + 0 \right) - \left(\frac{3\pi}{4} + 2 \right) \right] = \frac{1}{2}\left(\frac{3\pi}{4} - 2 \right) = \frac{1}{8} (3\pi - 8)$$

Soln. :

963

Let $I = \iint_R \frac{(x^2 + y^2)^2}{x^2 y^2} dx \cdot dy$

Let $x = r \cos \theta, y = r \sin \theta$;

the equation of circle is $r = a$.

$$\therefore \quad I = \int_0^{2\pi} \int_{r=0}^{a} \frac{(r^2)^2}{r^4 \cos^2 \theta \sin^2 \theta} r dr \, d\theta$$

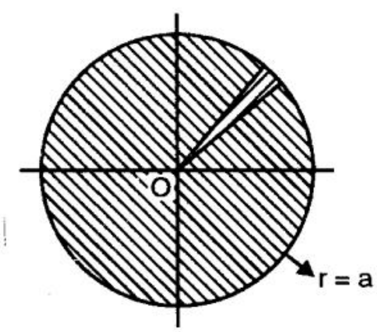


Fig. Ex. 9.37

$$= \int_0^{2\pi} \frac{1}{\cos^2\theta \sin^2\theta} d\theta \int_0^a r\,dr = 4 \int_0^{2\pi} \frac{1}{4\sin^2\theta\cos^2\theta} d\theta \left(\frac{r^2}{2}\right)_0^a$$

$$= 4 \int_0^{2\pi} \operatorname{cosec}^2 2\theta \cdot d\theta \left(\frac{a^2}{2}\right) = 2a^2 \int_0^{2\pi} \operatorname{cosec}^2 2\theta \cdot d\theta$$

Let $2\theta = t$

$$= 2a^2 \int_0^{4\pi} \operatorname{cosec}^2 t \cdot \frac{dt}{2} = a^2 [-\cot t]_0^{4\pi} = \infty$$

Soln. :

Q64)

Let $I = \int_0^a dx \int_{\sqrt{ax - x^2}}^{\sqrt{a^2 - x^2}} \frac{dy}{\sqrt{a^2 - x^2 - y^2}}$

limits are

$$y = \sqrt{ax - x^2} \text{ to } y = \sqrt{a^2 - x^2}$$

i.e., $x^2 + y^2 - ax = 0$, $\quad x^2 + y^2 = a^2$

and $x = 0$ to $x = a$.

Note that $x^2 + y^2 = a^2$ is a standard circle and $x^2 + y^2 - ax = 0$ is a circle touching Y-axis at origin and radius $\frac{a}{2}$.

The curve is as shown.

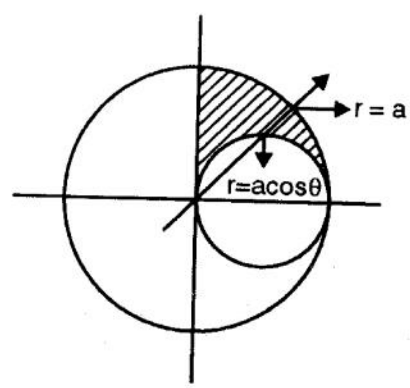


Fig. Ex. 9.43

Changing to polar :

Let $x = r\cos\theta$, $y = r\sin\theta$;

$\therefore$ Equation of $x^2 + y^2 = ax$ is $r = a\cos\theta$ and that of $x^2 + y^2 = a^2$ is $r = a$.

$$\therefore \quad I = \int_0^{\pi/2} \int_{a\cos\theta}^{a} \frac{r\,dr\,d\theta}{\sqrt{a^2 - r^2}}$$

$$= -\frac{1}{2} \int_0^{\pi/2} d\theta \int_{a\cos\theta}^{a} \frac{(-2r)\,dr}{\sqrt{a^2 - r^2}}$$

$$= -\frac{1}{2} \int_0^{\pi/2} d\theta \left[2\sqrt{a^2 - r^2} \right]_{a\cos\theta}^{a}$$

$$= -\int_0^{\pi/2} [0 - a\sin\theta]\,d\theta = a \int_0^{\pi/2} \sin\theta \cdot d\theta = a.$$

Sol. : (Q65)

$$I = \int_0^{\pi/4} \frac{1}{2}\left[-\frac{1}{1+r^2}\right]_0^{\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}\left[1-\frac{1}{1+\cos 2\theta}\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}\left(1-\frac{1}{2}\sec^2\theta\right) d\theta$$

$$= \frac{1}{2}\left[\theta-\frac{1}{2}\tan\theta\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right) = \frac{1}{8}(\pi-2).$$

Ex. 2 : Evaluate $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\, dr \cdot d\theta.$

Sol. :

$$I = \int_0^{\pi/2} -\frac{1}{3}\left[(a^2-r^2)^{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left[(a^2-a^2\cos^2\theta)^{3/2}-(a^2)^{3/2}\right] d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left[(a^2\sin^2\theta)^{3/2}-(a^2)^{3/2}\right] d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}(a^3\sin^3\theta-a^3)\, d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/2}(1-\sin^3\theta)\, d\theta$$

$$= \frac{a^3}{3}\left[\{\theta\}_0^{\pi/2}-\left\{\frac{2}{3}\cdot 1\right\}\right] \qquad \text{[By (B) page 1.26]}$$

$$= \frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right).$$

Sol. : The region of integration is the same as in the above example. Hence, putting $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$, we get

(966)

$$I = \int_0^{\pi/4} \int_0^{a/\cos\theta} \frac{r^2 \cos^2\theta}{r} \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^{a/\cos\theta} r^2 \cos^2\theta\,dr\,d\theta$$

$$= \int_0^{\pi/4} \left[\frac{r^3}{3}\right]_0^{a/\cos\theta} \cdot \cos^2\theta\,d\theta$$

$$= \int_\theta^{\pi/4} \frac{a^3}{3} \cdot \frac{1}{\cos^3\theta} \cdot \cos^2\theta\,d\theta = \frac{a^3}{3}\int_0^{\pi/4} \sec\theta\,d\theta$$

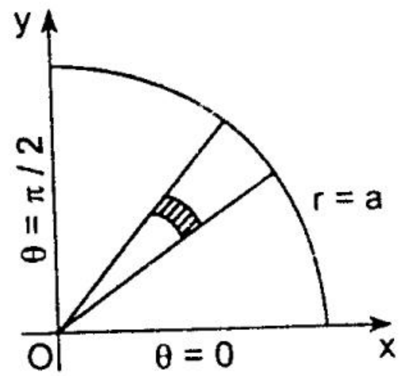
$$= \frac{a^3}{3}\left[\log(\sec\theta + \tan\theta)\right]_0^{\pi/4}$$

$$= \frac{a^3}{3}\left[\log(\sqrt{2} + 1) - \log 1\right]$$

$$= \frac{a^3}{3}\log\left(1 + \sqrt{2}\right).$$

(867) $\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dx\, dy.$ **(M.U. 1990, 2002)**

Sol. : Putting $x = r\cos\theta$, $y = r\sin\theta$ the given limit $y^2 = a^2 - x^2$ *i.e.* the circle $x^2 + y^2 = a^2$, changes to $r = a$ and $y = 0$ *i.e.* the x-axis changes to the initial line $\theta = 0$; $x = 0$, the y-axis becomes $\theta = \pi/2$. Hence, in the given region r changes from 0 to a and θ changes from 0 to $\pi/2$.



$$\therefore \quad I = \int_0^{\pi/2} \int_0^a e^{-r^2}\, r\, dr\, d\theta$$

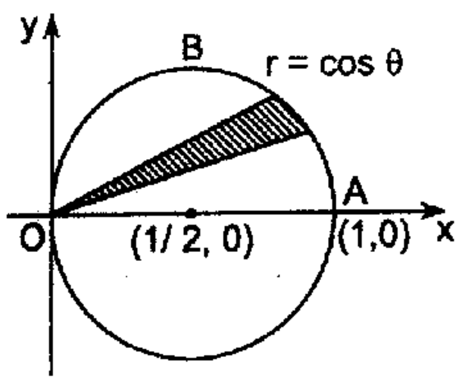
$$= \int_0^{\pi/2} \left[-\frac{1}{2} e^{-r^2} \right]_0^a d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} \left(e^{-a^2} - 1 \right) d\theta$$

$$= -\frac{1}{2} \left(e^{-a^2} - 1 \right) \left[\theta \right]_0^{\pi/2} = \frac{\pi}{4} \left(1 - e^{-a^2} \right).$$

(868) **(M.U. 2001)**

Sol. : The curve $x^2 + y^2 - x = 0$ *i.e.* $[x - (1/2)]^2 + y^2 = (1/2)^2$ is a circle with centre $[(1/2), 0]$ and radius $1/2$. The line $y = 0$ is the x-axis. The region of integration is the upper semi-circle *OAB*.



To change to polar put $x = r\cos\theta$, $y = r\sin\theta$. Then $x^2 + y^2 - x = 0$ changes to $r^2\cos^2\theta + r^2\sin^2\theta = r\cos\theta$ *i.e.* $r^2 = r\cos\theta$ *i.e.* $r = \cos\theta$.

Now, consider a radial strip. On this strip r varies from $r = 0$ to $r = \cos\theta$. Then θ varies from $\theta = 0$ to $\theta = \pi/2$.

$$\therefore \quad I = \int_0^{\pi/2}\int_0^{\cos\theta} \frac{1}{r\sqrt{\sin\theta\cos\theta}}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{\cos\theta} \frac{dr\, d\theta}{\sqrt{\sin\theta\cos\theta}} = \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta\cos\theta}}\left[r\right]_0^{\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta\cos\theta}} \cdot \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta\, \cos^{1/2}\theta\, d\theta$$

$$= \frac{1}{2} B\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{1}{2} \cdot \frac{\overline{|1/4}\ \overline{|3/4}}{\overline{|1}}$$

$$= \frac{1}{2} \cdot \sqrt{2} \cdot \pi = \frac{\pi}{\sqrt{2}}.$$

7)

$$I = \int_{-a}^{a} \int_{-(b/a)\sqrt{a^2-x^2}}^{+(b/a)\sqrt{a^2-x^2}} (x^2 + 2xy + y^2)\, dx\, dy$$

$$= \int_{-a}^{a} \left[x^2 y + xy^2 + \frac{y^3}{3} \right]_{-(b/a)\sqrt{a^2-x^2}}^{+(b/a)\sqrt{a^2-x^2}}$$

$$= \int_{-a}^{a} 2 \left\{ x^2 \frac{b}{a} \sqrt{a^2 - x^2} + \frac{b^3}{3a^3} \sqrt{(a^2 - x^2)^3} \right\} dx$$

Put $x = a \sin\theta \quad \therefore \quad dx = a \cos\theta \, d\theta$

$$I = 2\int_{-\pi/2}^{\pi/2} \left\{ a^2 \sin^2\theta \cdot \frac{b}{a} \cdot a\cos\theta + \frac{b^3}{3a^3} a^3 \cos^3\theta \right\} a\cos\theta \; d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \left\{ a^3 b \sin^2\theta \cos^2\theta + \frac{1}{3} ab^3 \cos^4\theta \right\} d\theta$$

$$= 4\int_{0}^{\pi/2} \left\{ a^3 b \sin^2\theta \cos^2\theta + \frac{1}{3} ab^3 \cos^4\theta \right\} d\theta$$

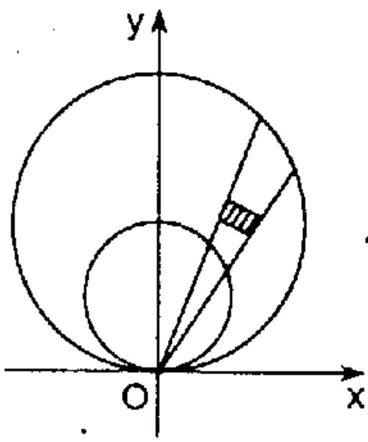
$$= 4\left[a^3 b \cdot \frac{1}{2} \cdot \frac{\overline{|3/2}\ \overline{|3/2}}{\overline{|3}} + \frac{1}{3} ab^3 \cdot \frac{1}{2} \cdot \frac{\overline{|1/2}\ \overline{|5/2}}{\overline{|3}} \right]$$

$$= 4\left[a^3 b \cdot \frac{1}{2} \cdot \frac{\left\{ (1/2)\overline{|1/2} \right\}^2}{2} + \frac{1}{3} ab^3 \cdot \frac{1}{2} \cdot \frac{\overline{|1/2} \cdot (3/2) \cdot (1/2) \cdot \overline{|1/2}}{2} \right]$$

$$= 4\left[\frac{a^3 b \cdot \pi}{16} + \frac{ab^3}{16}\pi \right] = \frac{\pi ab}{4}\left[a^2 + b^2 \right].$$

(074) **Sol. :** The circle $r = 2\sin\theta$ i.e. $r^2 = 2r\sin\theta$ becomes in cartesian system $x^2 + y^2 = 2y$ i.e. $x^2 + (y-1)^2 = 1$. Similarly, the circle $r = 4\sin\theta$ i.e.

$r^2 = 4r\sin\theta$ becomes in cartesian system $x^2 + y^2 = 4y$ i.e. $x^2 + (y-2)^2 = 4$. In the given region r varies from $2\sin\theta$ to $4\sin\theta$ and θ varies from 0 to π.



$$\therefore \quad I = \int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3\, dr\, d\theta$$

$$= \int_0^{\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi}\left[4^4\sin^4\theta - 2^4\sin^4\theta\right] d\theta$$

$$= 60\int_0^{\pi}\sin^4\theta\, d\theta$$

$$= 120\int_0^{\pi/2}\sin^4\theta\, d\theta = 120\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{45}{2}\cdot\pi.$$